Exam 1 **Calculus 3** **9/28/18**

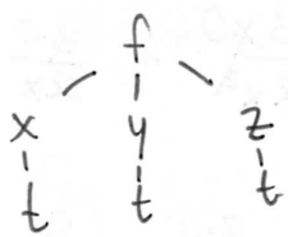
Each problem is worth 10 points. Full credit requires good justification for your answers.

1. State the definition of the partial derivative of $f(x, y)$ with respect to y.

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

Good

2. Suppose that f is a function of x, y, and z, each of which is a function of t. Write the Chain Rule formula for $\frac{df}{dt}$, and make very clear which derivatives are partials.



Great

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

partial partial partial

3. Write an equation for the plane tangent to $f(x, y) = x^2 + 3xy + 2y^2 + 6y - 4$ at the point (1,-2).

$f_x(x,y) = 2x + 3y \quad f_x(1,-2) = 2(1) + 3(-2) = 2 - 6 = -4$

$f_y(x,y) = 3x + 4y + 6 \quad f_y(1,-2) = 3(1) + 4(-2) + 6 = 3 + -8 + 6 = 1$

$f(1,-2) = 1^2 + 3(1)(-2) + 2(-2)^2 + 6(-2) - 4$

$= 1 + -6 + 8 + -12 - 4 = -13$

$\underline{z + 13 = -4(x-1) + 1(y+2)}$ Great

4. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy}{2x^2 + y^2}$ does not exist.

$z_0 = -13$

Approach on $\underline{x = 0}$

$$\lim_{(0,y)\to(0,0)} \frac{3(0)y}{2(0)^2 + y^2} = \lim_{(0,y)\to(0,0)} \frac{0}{y^2} = \underline{0}$$

Approach on $\underline{x = y}$

$0 \neq 1$

$$\lim_{(x,x)\to(0,0)} \frac{3xx}{2x^2 + x^2} = \lim_{(x,x)\to(0,0)} \frac{3x^2}{3x^2} = \underline{1}$$

$\therefore$ the limits from two different directions are not equal, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy}{2x^2+y^2} = \underline{DNE}$

Great

5. (a) Find the directional derivative of the function $g(p,q) = p^4 - p^2q^3$ at the point $(1,2)$ in the direction of the vector $\vec{v} = \vec{i} - 3\vec{j}$.

$D_{\vec{u}}(p_0, q_0) = a \cdot g_p(p_0, q_0) + b \cdot g_q(p_0, q_0)$

$g_p(p,q) = 4p^3 - 2pq^3$ $\quad g_p(1,2) = 4(1)^3 - 2(1)(2)^3 = 4 - 16 = \underline{-12}$

$g_q(p,q) = -3p^2q^2$ $\quad g_q(1,2) = -3(1)^2(2)^2 = \underline{-12}$

$\vec{v} = i - 3j$ $\quad |\vec{v}| = \sqrt{(1)^2 + (-3)^2} = \sqrt{1+9} = \sqrt{10}$ $\quad \underline{a = \frac{1}{\sqrt{10}}}, \underline{b = \frac{-3}{\sqrt{10}}}$

$D_{\vec{u}}(p_0, q_0) = \frac{1}{\sqrt{10}} \cdot (-12) + \left(\frac{-3}{\sqrt{10}}\right) \cdot (-12) = \frac{-12}{\sqrt{10}} + \frac{36}{\sqrt{10}} = \underline{\frac{24}{\sqrt{10}}}$

(b) In which direction is the directional derivative greatest at the point (1,2)?

The gradient of $g(p,q)$ at point $(1,2)$ gives the direction of $D_{\vec{u}}(p_0,q_0)$'s max.

$\nabla g(1,2) = \langle g_p(1,2), g_q(1,2) \rangle = \underline{\langle -12, -12 \rangle}$

Excellent!

6. Show that for any vectors $\vec{a}$ and $\vec{b}$, the vector $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$.

Well, I know if the dot product of two vectors equals zero, then they must be perpendicular.

$$\vec{a} = \langle a_1, a_2, a_3 \rangle$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3i - a_3b_2i) + (a_3b_1j - a_1b_3j) + (a_1b_2k - a_2b_1k)$$

$$\vec{a} \times \vec{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

$$\vec{a} \cdot (\vec{a} \times \vec{b}) = \cancel{a_1a_2b_3} - \cancel{a_1a_3b_2} + \cancel{a_2a_3b_1} - \cancel{a_2a_1b_3} + \cancel{a_3a_1b_2} - \cancel{a_3a_2b_1}$$

$$\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$$

So, because $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$, $\vec{a}$ must be perpendicular to $\vec{a} \times \vec{b}$

Great!

7. Bunny is a Calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod, Calc is so impossible! Even stuff that I thought made sense and was okay gets so confusing for what you actually need to be able to do, you know? Like, our professor was saying we wouldn't have to be able to draw the level curve diagrams, right? Because it's so hard to do it good by hand, right? But then he said it would be fair if it was one like for a plane, and now I'm totally panicing, because I have no idea what that would be like. Should I just drop and change my major?"

Help Bunny out by explaining clearly what can be said about the level curves of a plane.

hey Bunny, its okay, because level curve diagrams still kinda confuse me too. Its fine for shaded diagrams but as soon as they take away the shading and give values ... yeah, I getcha.

But lucky you! Level curves of planes are really quite easy -- we could draw one together, without any equation necessary.

So a plane, hanging out in 3-D space doesn't have any humps or valleys -- its a flat surface, right?

So if we hold up a book to use as a plane, we can tilt that book all over the place, but since its not curved in any way, the level curves on that book are straight lines. So the curve diagram would look like a bunch of stripes, with the highest corner having the highest value, and the lowest corner having the lowest value:

Wow!

8. Find the minimum value of the function $f(x, y) = x^2 + y^2$ subject to the constraint $x + 2y = 6$.

Using Lagrange $\nabla f = \lambda \cdot \nabla g$

$\nabla f = \langle 2x, 2y \rangle$ $\nabla g = \langle 1, 2 \rangle$

$$\begin{array}{r} 12 \\ \times 12 \\ \hline 24 \\ 12 \\ \hline 144 \end{array}$$

$\langle 2x, 2y \rangle = \lambda \cdot \langle 1, 2 \rangle$

$\lambda = 2x$ and $\cancel{2}\lambda = \cancel{2}y$

$\lambda = y$

$\lambda =$ $\boxed{y = 2x}$ plugging into $x + 2y = 6$

$x + 2(2x) = 6$

$x + 4x = 6$

$5x = 6$

$x = 6/5$ $y = 2(6/5) = 12/5$

Point $(6/5, 12/5)$

If one point we trust,

Minimum value $= f(6/5, 12/5) = \left(\frac{6}{5}\right)^2 + \left(\frac{12}{5}\right)^2$

$= \frac{36}{25} + \frac{144}{25} = \boxed{180/25}$

Well done

9. Find all critical points of $f(x, y) = x^3 - 6xy + 8y^3$ and classify them as maxima, minima, or saddle points.

$f_x(x,y) = 3x^2 - 6y = 0$

$3x^2 = 6y$

$x^2 = 2y$

$f_y(x,y) = 24y^2 - 6x = 0$

$24y^2 = 6x$

$4y^2 = x$

$(2y)^2 = x$

$(x^2)^2 = x$

$x^4 = x$

$x^4 - x = 0$

$x(x^3 - 1) = 0$

$x = 0 \qquad x^3 - 1 = 0$

$x^3 = 1$

$x = 1$

$x = 0$

$(0)^2 = 2y$

$0 = 2y$

$0 = y$

$x = 1$

$1^2 = 2y$

$1 = 2y$

$\frac{1}{2} = y$

$f_{xx} = 6x$

$f_{yy} = 48y$

$f_{xy} = -6$

$D(x,y) = 6x \cdot 48y - (-6)^2$

$D(x,y) = 288xy - 36$

$D(0,0) = 288(0)(0) - 36 = -36 < 0$

$\therefore$ $(0,0)$ is a saddle

$D(1, \frac{1}{2}) = 288(1) \cdot (\frac{1}{2}) - 36 = 108 > 0$

$f_{xx}(1, \frac{1}{2}) = 6(1) = 6 > 0$

$\therefore$ $(1, \frac{1}{2})$ is a min

Critical Pts.

$(0,0)$ saddle pt.

$(1, \frac{1}{2})$ minima

Excellent!

10. Find an equation for the plane tangent to the surface $x^2 + y^2 + xz + z^2 = 4$ at the point (1,1,1).

$$f(x,y,z) = x^2 + y^2 + xz + z^2$$

$$\nabla f(x,y,z) = \langle 2x+z,\ 2y,\ x+2z \rangle$$

gradient $\perp$ to level curves at a pt.

$$\nabla f(1,1,1) = \langle 2(1)+1,\ 2(1),\ 1+2(1) \rangle$$

$$= \langle 3, 2, 3 \rangle$$

$$0 = -f_x(x-x_0) - f_y(y-y_0) - f_z(z-z_0)$$

$$f_x(x-x_0) + f_y(y-y_0) + f_z(z-z_0) = 0$$

$$3(x-1) + 2(y-1) + 3(z-1) = 0$$

$$3x - 3 + 2y - 2 + 3z - 3 = 0$$

$$\boxed{3x + 2y + 3z = 8}$$

Great