Exam 2 Calc 1 10/4/2019

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the formal definition of the derivative of a function $f(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Good

2. Find an equation of the tangent line to the curve $y = \cos x - \sin x$ at the point $(\pi, -1)$.

$y' = (\cos x)' - (\sin x)'$

$y' = -\sin x - \cos x$

$y' = -\sin(\pi) - \cos(\pi)$

$y' = -0 - (-1) = +1 = 1 = m$

$y - y_1 = m(x - x_1)$

$\boxed{y + 1 = x - \pi}$

Good

3. Prove the Sum Rule for Derivatives, that if $f(x)$ and $g(x)$ are two differentiable functions then $(f+g)'(x) = f'(x) + g'(x)$.

If $f(x)$ and $g(x)$ are 2 differentiable functions

$$(f+g)'(a) = \lim_{x \to a} \frac{(f+g)(x) - (f+g)(a)}{x-a}$$

$$= \lim_{x \to a} \frac{f(x) + g(x) - f(a) - g(a)}{x-a}$$

$$= \lim_{x \to a} \left[\frac{f(x) - f(a)}{x-a} + \frac{g(x) - g(a)}{x-a} \right]$$

$$= f'(a) + g'(a)$$

$\rightarrow (f+g)'(x) = f'(x) + g'(x)$ [Prove !]

Good.

4. A table of values for f, g, f', and g' is given below.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	–1	3	5	4
2	6	7	2	5
3	9	3	1	7

a) If $h(x) = f(x) \cdot g(x)$, what is $h'(3)$ and why?

product rule

$f'(x) \cdot g(x) + g'(x) \cdot f(x)$

$1 \cdot 3 + 7 \cdot 9$

$3 + 63$

66

b) If $h(x) = f(x) / g(x)$, what is $h'(2)$ and why?

Quotient rule

$2 \cdot 7 - 6 \cdot 5$

$$\frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x)^2}$$

7^2

$$-\frac{16}{49}$$

c) If $h(x) = f(g(x))$, what is $h'(1)$ and why?

Chain rule

Good

$f'(g(x)) \cdot g'(x)$

$f'(3) \cdot 4$

$1 \cdot 4$

4

5. Differentiate

a) $f(x) = x^3 \cos x$ product rule

$f'(x) = 3x^2(\cos x) + (x^3)(-\sin x)$

$f'(x) = 3x^2 \cos(x) - x^3 \sin(x)$

b) $g(x) = \dfrac{\cos x}{x^3}$ Quotient

$$\frac{(-\sin(x))(x^3) - (\cos x)(3x^2)}{(x^3)^2}$$

$$g'(x) = \frac{-x^3 \sin(x) - 3x^2 \cos(x)}{(x^3)^2}$$

c) $h(x) = \cos(x^3)$ Chain rule

$-\sin(x^3) \cdot 3x^2$

$h'(x) = -3x^2 \sin(x^3)$

Excellent

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "*Ohmygod.* This Calculus stuff is so unfair! I swear, it's literally the least fair thing ever. There was this one function, like the cube root, right, and like, the professor was talking about it right at the end of class because somebody asked a question, right? And the question was about the tangent line when it's zero, but like, somehow the calculator said error, right? So the professor said there *is* a tangent line, but it's not wrong that the calculator said error, which is totally contradictory and unfair, but class was ending so there were, like, 200 people standing up in front of me and I have no idea what he was saying, so now I'm going to *fail!*"

   Help Bunny by explaining as clearly as possible why a calculator might get an error in connection with such a question, but the tangent line might still exist.

Oh my god, so like, when you plug a zero into this function's derivative, the calculator says error because it's probably something like a zero in the denominator, which doesn't exist, right? So like that means that there's totally no limit to this function when it's zero. But like regular lines can be straight up and down, right? And like, a tangent line is basically a regular line when you think about it, right? so if the tangent line is vertical, it obviously exists, but for some reason, probably like because the limit doesn't exist at a vertical line, your calculator can't handle it and fails to compute, unlike you now, Bunny!

She gets it!

8. Use the definition of the derivative to find the derivative of $f(x)=\frac{1}{x}$.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\frac{1}{x} - \frac{1}{a}}{x - a}$$

$$= \lim_{x \to a} \frac{a - x}{ax} \times \frac{1}{x - a}$$

$$= \lim_{x \to a} \frac{-(x - \cancel{a})}{ax} \times \frac{1}{(\cancel{x - a})}$$

$$= \lim_{x \to a} -\frac{1}{ax}$$

$$= \frac{-1}{a^2}$$

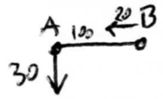
for $f'(x)$,

when, $a = x$

$$\boxed{f'(x) = \frac{-1}{x^2}}$$

well done

9. At noon, ship A is 100 km west of ship B. Ship A is sailing south at 30 km/h and ship B is sailing west at 20 km/h. How fast (to the nearest tenth of a mile per hour) is the distance between them changing at 2:00 PM?



$\Delta t = 2$

Ship A starts at origin

$P_a = 30 \cdot 2 = 60$

$P_b = 100 - 20 \cdot 2 = 60$

$P_c = \sqrt{60^2 + 60^2} = 84.8528$

$a^2 + b^2 = c^2$

$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$

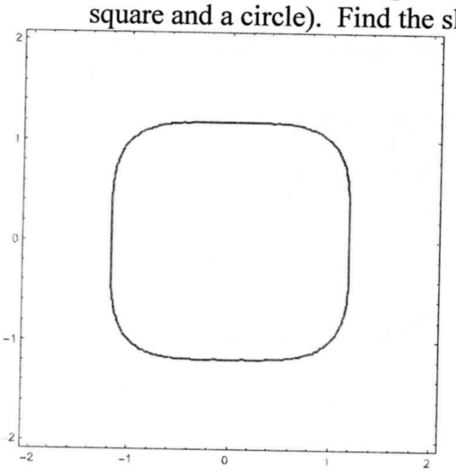
$2 \cdot 60 \cdot 30 + 2 \cdot 60 \cdot -20 = 2 \cdot 84.8528 \cdot \frac{dc}{dt}$

$$\frac{dc}{dt} = \frac{2 \cdot 60 \cdot 30 + 2 \cdot 60 \cdot -20}{2 \cdot 84.8528} = 7.07$$

$$= \boxed{7.1}$$

Excellent

10. a) The curve with implicit equation $x^4 + y^4 = 2$ is sometimes called a squircle (like a mix of a square and a circle). Find the slope of the tangent line to the squircle at the point $(1, -1)$.



$$x^4 + y^4 = 2$$

$$4x^3 + 4y^3\frac{dy}{dx} = 0$$

$$4y^3\frac{dy}{dx} = -4x^3$$

$$\frac{dy}{dx} = \frac{-x^3}{y^3}$$

When $x = 1$, $y = -1$,

Nice!

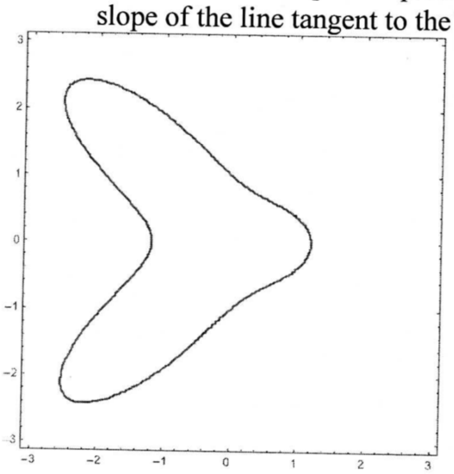
$$\frac{dy}{dx} = m = \frac{-(1)^3}{(-1)^3} = \underline{1}$$

$$y+1 = 1(x-1)$$

$$y+1 = x-1$$

$$\boxed{y = x-2}$$

b) The curve with implicit equation $x^4 + 7xy^2 + 2y^4 = 2$ is hereby called a blorg. What is the slope of the line tangent to the blorg at $(0,1)$?



$$x^4 + 7xy^2 + 2y^4 = 2$$

$$4x^3 + 7\left[y^2 + x\cdot 2y\frac{dy}{dx}\right] + 2\times 4y^3\frac{dy}{dx} = 0$$

$$4x^3 + 7y^2 + 14xy\frac{dy}{dx} + 8y^3\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(14xy + 8y^3\right) = -4x^3 - 7y^2$$

$$\frac{dy}{dx} = \frac{-4x^3 - 7y^2}{14xy + 8y^3}$$

Excellent

When $x = 0$, $y = 1$

$$\frac{dy}{dx} = m = \frac{0 - 7\times 1}{0 + 8} = \frac{-7}{8}$$

$$y - 1 = \frac{-7}{8}(x - 0)$$

$$\boxed{y = \frac{-7}{8}x + 1}$$