Exam 3 Calc 1 10/23/2019

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. What is $(e^x)'$?

$(e^x)' = e^x$

At any point on the graph of e^x, the height of the point is equal to the derivative at that point

Cool.

2. a) What is $(10^x)'$?

$10^x \ln(10)$

Good

b) What is $(\log_2 x)'$?

$\frac{1}{x \ln 2}$

3. Evaluate $\lim_{x\to\infty} \frac{x}{e^x}$. Be sure to provide good justifications for your steps.

$\lim_{x\to\infty} \frac{x}{e^x}$ which is basically $\lim_{x\to\infty} \frac{\infty}{\infty}$

so use L'H rule

Great!

$\lim_{x\to\infty} \frac{x}{e^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{1}{e^x}$ which is $\frac{1}{\text{big\#}}$ so $\lim_{x\to\infty} \frac{1}{e^x} = \emptyset$

4. a) If $f(x) = x^3 \ln x$, what is $f'(x)$?

$f(x) = (x^3)(\ln x)$

use product rule!

$f'(x) = 3x^2(\ln x) + x^3\left(\frac{1}{x}\right)$

$= \boxed{3x^2(\ln x) + x^2}$

Excellent!

b) If $g(x) = \ln(x^3)$, what is $g'(x)$?

$g(x) = \ln(x^3)$

use chain rule!

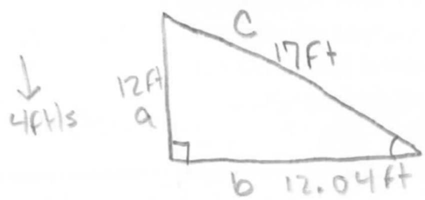
$g'(x) = \frac{1}{x^3} \cdot 3x^2$

$= \frac{3x^2}{x^3}$

$= \boxed{\frac{3}{x}}$

5. A 17 foot ladder is leaning against a wall. If the top slips down the wall at a rate of 4 ft/s, how fast will the foot be moving away from the wall when the top is 12 feet above the ground?

$$a^2+b^2=c^2$$



$$b=\sqrt{17^2-12^2}$$

$$b=12.04\text{ft}$$

Take derivative

$a=12$

$b=12.04$

$c=17$

$a'=-4$

$b'=?$

$c'=0$

$$a^2+b^2=c^2$$

$$2a\cdot a'+2b\cdot b'=2c\cdot c'$$

$$2(12)(-4)+2(12.04)(b')=2(17)(0)$$

$$-96+24.08b'=0$$

$$24.08b'=96$$

$$b'=3.9862\text{ ft/s}$$

The foot will be moving away from the wall at a rate of $\boxed{3.9862\text{ ft/s}}$

Excellent

6. Show why the derivative of arctan x is what it is.

I Know:

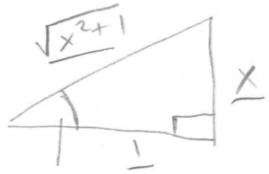
$$\tan(\tan^{-1}x) = x$$

Differentiate!

$$\sec^2(\arctan x) \cdot (\arctan x)' = 1$$

Chain rule

$$(\arctan x)' = \frac{1}{\sec^2(\arctan x)}$$



I'm using this angle for arctan x and to find two side lengths whose ratio is x.

$$(\arctan x)' = \frac{1}{(\sqrt{x^2+1})^2}$$

Found this by knowing that sec is hypotenuse over adjacent

$$(\arctan x)' = \frac{1}{x^2+1}$$

Now with pythagorean theorem I find my third side

$$x^2 + 1^2 = c^2$$

$$x^2 + 1 = c^2$$

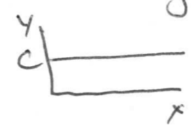
$$\sqrt{x^2+1} = c$$

Excellent!

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Geez, calculus is hard! I though I had pretty much the perfect plan, you know? I've got, you know, one of those calculators that does derivatives for you, right? So I thought the exam would be really easy, but it totally didn't work. There was this one batch of questions on the exam with all these functions to do derivatives of, right, but it was the log of 7, and then like the inverse sine of one half, and *e* to the seven or something, right? But the calculator said it was zero for all of them, which is totally stupid since they're not even the same function, so they can't have the same derivative, right?"

Explain clearly to Biff what's going on.

Biff's problem here is that he inputs all those numbers which are clearly constant into the calculators and then find the derivatives. As we all know, $\log 7$; $\sin^{-1}\left(\frac{1}{2}\right)$ and e^7 are all constant numbers because they do not include any variables. Derivatives are basically the rate change of a function. So when we have functions like $y = \text{Constant}$, the graph will look like:



and there's clearly no rates of change. That is why derivatives of constant $\left(\frac{dC}{dx}\right)$ always 0.

Excellent

8. A research sample of 1000 zombie-spawning bacteria escapes the lab and begins to grow at a rate proportional to its population. After 5 hours there are 3000 bacteria. Give a formula for the number of bacteria after t hours.

solve A

$P(t) = Ae^{k(t)}$

$P(0) = Ae^{k(0)}$

$1{,}000 = A$

solve K

$P(t) = 1{,}000e^{(k)(t)}$

$P(5) = 1{,}000e^{(k)(5)}$

$\frac{3{,}000}{1{,}000} = \frac{1{,}000}{1{,}000}e^{(k)(5)}$

$3 = e^{(k)(5)}$

Natural log both sides to get rid of e

$\frac{\ln(3)}{5} = \frac{5k}{5}$

$k = \frac{\ln(3)}{5}$

Well done!

plug all variables back in

$P(t) = 1{,}000e^{\left(\frac{\ln(3)}{5}\right)(t)}$

9. A freshly brewed cup of coffee sits on a counter in a 21°C room at 95°C. After 5 minutes its temperature is down to 85°C. How soon will it be 60°C? [Hint: the general solution to Newton's Law of Cooling is $T(t) = T_s + Ae^{kt}$, where T_s is the temperature of the surrounding medium.]

$T(t) = T_s + Ae^{kt}$

At $t = 0$, $T(t) = 95°C$

$95 = 21 + Ae^{k \times 0}$

$95 = 21 + A$

$A = 74$

When $t = 5min$, $T(t) = 85°C$

$85 = 21 + 74e^{k \times 5}$

$64 = 74e^{5k}$

$e^{5k} = \frac{64}{74}$

Taking ln to both sides:

$5k \ln(e) = \ln\left(\frac{64}{74}\right)$

$5k = \ln\left(\frac{64}{74}\right)$

$k = \ln\left(\frac{64}{74}\right) / 5$

$k = -0.02904$

$\therefore T(t) = 21 + 74e^{-0.2904 \times t}$

When $T(t) = 60$:

$60 = 21 + 74e^{-0.2904 \times t}$

$39 = 74e^{-0.2904 \times t}$

$\frac{39}{74} = e^{-0.2904 \times t}$

Taking ln to both sides:

$\ln(39/74) = -0.2904t \ln(e) \to 0$

$t = \ln\left(\frac{39}{74}\right) / -0.2904$

$\boxed{t = 22.058}$

Great

$-x^{-2}$

x^{-1}

$-1x^{-2}$

10. Evaluate $\lim_{x\to\infty}\dfrac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$. Be sure to provide good justifications for your steps.

$$\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)\to 0}{\frac{1}{x}\to 0} \overset{L'H}{=} \lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\cdot\left(0+\frac{-1}{x^2}\right)}{\frac{-1}{x^2}} =$$

$$\lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\cdot\cancel{\frac{-1}{x^2}}}{\cancel{\frac{-1}{x^2}}} = \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = \frac{1}{1+0} = \frac{1}{1} = \boxed{1}$$

Wonderful!

$$\lim_{x\to\infty}\frac{\left(\ln\left(1+\frac{1}{x}\right)\right)}{\frac{1}{x}} = 1$$