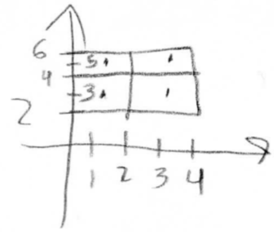
Exam 2 Calc 3 10/25/2019

Each problem is worth 10 points. For full credit provide complete justification for your answers. All integrals should be set up in terms of a single coordinate system, i.e., if you use cylindrical your integral should involve no x or y, etc.

W

1. Write a double Riemann sum for $\iint_R f\,dA$, where $R = \{(x, y) : 0 \le x \le 4, 2 \le y \le 6\}$ using midpoints with $n = m = 2$ subdivisions.

Midpoints: (1,3), (1,5), (3,3), (3,5)

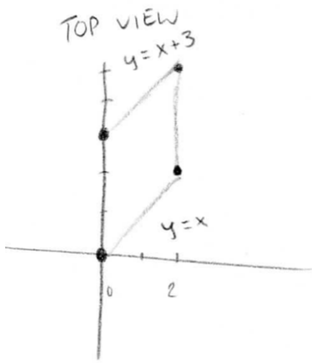


$$\boxed{\iint_R f\,dA \approx 4\big(f(1,3) + f(1,5) + f(3,3) + f(3,5)\big)}$$

Good

$\Delta y = 2$
$\Delta x = 2$
$\Delta A = 4$

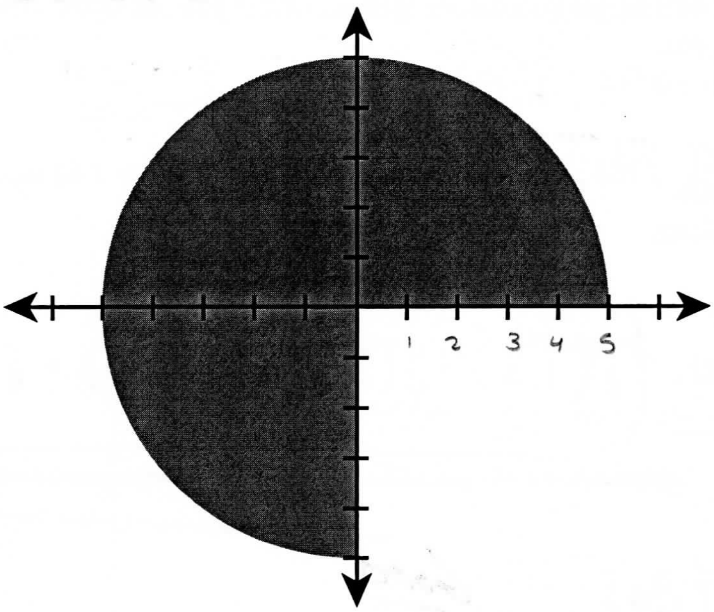
2. Let $f(x,y) = 4x^2 + 9y^2$. Let R be the parallelogram with vertices (0,0), (2,2), (2,5), and (0,3). Set up an iterated integral for $\iint_R f\,dA$.



$$\int_0^2 \int_x^{x+3} 4x^2 + 9y^2 \cdot dy\,dx$$

Good

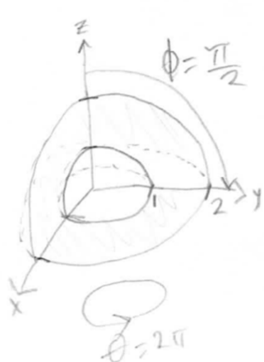
3. Set up an iterated integral for the total mass of a plate shaped like the region shown below, with density $\rho(x, y) = k$.



$$* \int_0^{\frac{3\pi}{2}} \int_0^5 k r \, dr \, d\theta$$

Good

4. Set up limits of integration for an iterated integral for the volume of the region above the *xy*-plane, outside a sphere of radius 1 centered an the origin, and inside a sphere of radius 2 centered at the origin.



Spherical

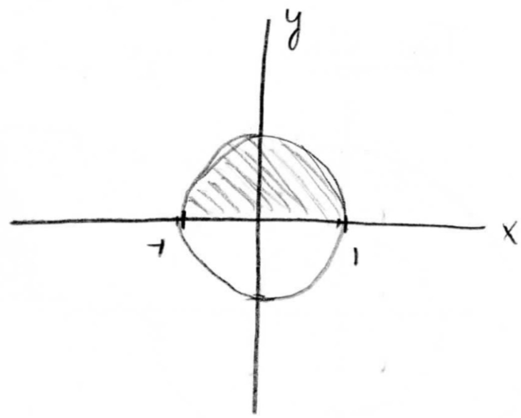
$$\int_0^{2\pi} \int_0^{\pi/2} \int_1^2 1 \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Great

5. Evaluate $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} 8\,dy\,dx$.

$y = \sqrt{1-x^2}$

↳ circle w/ radius 1



top half of a circle with radius 1

$= 8 \cdot \frac{1}{2}\pi r^2$

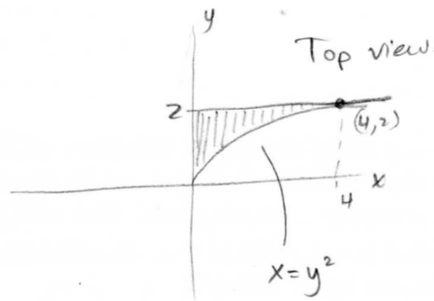
$A_{circle} = \pi r^2$?

$= 8 \cdot \frac{1}{2}\pi(1)^2$

$= 4\pi$

Excellent!

6. Evaluate $\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{y^3+1}\,dy\,dx$.



Nice!

Change order of integration:

$$= \underline{\int_0^2 \int_0^{y^2} \frac{1}{y^3+1}\,dx\,dy}$$

$$= \int_0^2 \left[\frac{x}{y^3+1}\right]_0^{y^2} dy$$

$$= \int_0^2 \left(\frac{y^2}{y^3+1}\right) dy$$

Now let $u = y^3+1$.

$\frac{du}{3} = \cancel{3}y^2\,dy$

$$= \frac{1}{3}\int_0^2 \frac{1}{u}\,du = \left[\frac{1}{3}\ln\left(y^3+1\right)\right]_0^2$$

$$= \frac{1}{3}\left(\ln(9) - \cancelto{0}{\ln(1)}\right)$$

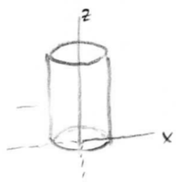
$$= \boxed{\frac{\ln(9)}{3}}$$

Excellent!

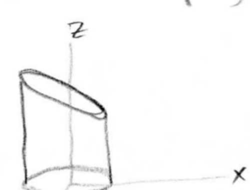
7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "OMG! Calc 3 is just so much! It's like, these things I thought I understood before are more confusing now, like with polar? So like sometimes you go zero to 2 pi, but sometimes you go 0 to pi over two and take it times four, but sometimes you can't do that. How do you know which?"

Explain clearly to Bunny an example where her two ways are equal, and also an example where they are not, and why.

Bunny, it depends on symmetry. For example, if you have a cylinder with a flat top like this



so the top view is a circle, you can just go from 0 to $\frac{\pi}{2}$ and multiply by 4 because you know all four parts are the same. But if the top isn't flat, like it has a sloping roof or something, like this



then you know the four pieces of it are different + you'd better go from 0 to 2π. Here's a tip: always look at the side view.

Excellent!

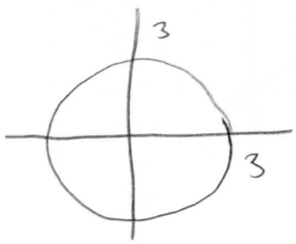
$z = 9 - r$

$r = 9 - z$

8. Set up iterated integrals for the centroid of the solid bounded by $z = 9 - \sqrt{x^2 + y^2}$ above the plane $z = 0$ and below the plane $z = b$ for some value $0 \le b \le 9$.

cone

Top view



$\rho(x,y) = k$

$0 = 9 - \sqrt{x^2+y^2}$

$r = 3$

$b = 9 - r$

$r = 9 - b$

$$V = \int_0^{2\pi}\int_0^{b}\int_0^{9-z} k\, r\, dr\, dz\, d\theta$$

$\bar{x}$ and $\bar{y}$ are 0 by symmetry since it's centered at the origin of the x-y axiz

$$\boxed{\bar{z} = \frac{\int_0^{2\pi}\int_0^{b}\int_0^{9-z} k z r\, dr\, dz\, d\theta}{\int_0^{2\pi}\int_0^{b}\int_0^{9-z} k r\, dr\, dz\, d\theta}}$$

Excellent!

9. Find the Jacobian for the transformation from rectangular to spherical coordinates.

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ \rho\cos\phi\cos\theta & \rho\cos\phi\sin\theta & -\rho\sin\phi \\ -\rho\sin\phi\sin\theta & \rho\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= (\rho^2 \sin^3\phi \cos^2\theta) + (\rho^2 \sin^3\phi \sin^2\theta) + (\rho^2 \sin\phi \cos^2\phi \cos^2\theta + \rho^2 \sin\phi \cos^2\phi \sin^2\theta)$$

$$= \rho^2 \sin\phi \sin^2\phi (\cos^2\theta + \sin^2\theta) + \rho^2 \sin\phi \cos^2\phi (\cos^2\theta + \sin^2\theta)$$

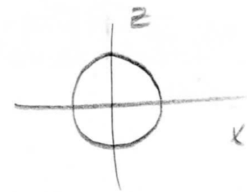
$$= \rho^2 \sin\phi (\sin^2\phi + \cos^2\phi)$$

$$= \rho^2 \sin\phi$$

Nice

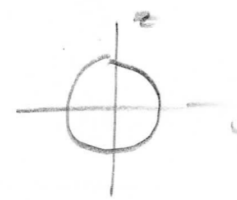
$$\boxed{\text{Jacobian for rectangular to spherical coordinates} = \rho^2 \sin\phi}$$

10. Set up one or more iterated integrals for the volume of the region inside both the cylinder $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$.



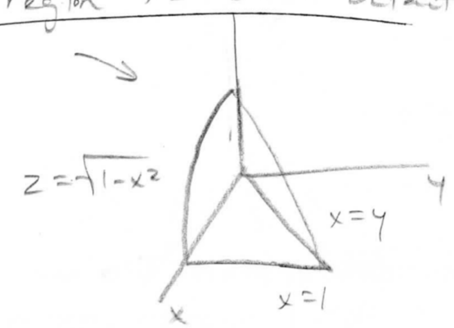
$$16 \int_0^1 \int_y^1 \int_0^{\sqrt{1-x^2}} 1 \, dz\,dx\,dy$$

Nice Job.

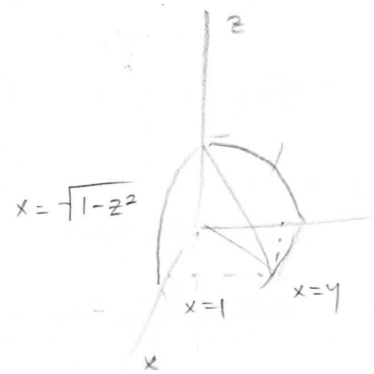


this region x2 each octant

= x16



Lovely!



$$x^2 + z^2 = y^2 + z^2$$

$$x^2 = y^2$$

$$x = y$$