Exam 3 **Foundations of Advanced Math** **10/25/19**

1. The sum of two bounded functions, both with domain $\mathbb{R}$, is bounded.

Proof: A function f is bounded $\ni \exists M_1 \in \mathbb{R} \ni \forall x, |f(x)| \leq M_1$.

A function g is bounded $\ni \exists M_2 \in \mathbb{R}, \forall x, |g(x)| \leq M_2$.

$(f+g)(x) = f(x) + g(x)$

$|f(x)| + |g(x)| \leq M_1 + M_2$

$|(f+g)(x)| \leq |f(x)| + |g(x)| \leq M_1 + M_2$ by the Triangle Inequality

$\therefore$ $f+g$ is bounded. $\square$

Nice!

2. If $f : A \to B$ and $g : B \to C$ are injective functions, then $g \circ f$ is injective.

$g \circ f(a_1) = g \circ f(a_2) \rightarrow g(f(a_1)) = g(f(a_2))$ Composition of functions

$g(f(a_1)) = g(f(a_2)) \rightarrow f(a_1) = f(a_2)$ g is injective

$f(a_1) = f(a_2) \rightarrow a_1 = a_2$ f is injective

$\therefore$ $g \circ f$ is injective

Good

3. Let $f : A \to B$ be an invertible function. Then f is bijective.

Well, let $f: A \to B$ be invertible, so $\exists g: B \to A$ such that $\forall a \in A, b \in B$

$f \circ g(b) = b$ and $g \circ f(a) = a$.

Now s'pose $f(a_1) = f(a_2)$. Then applying g to these equal inputs gives us $g \circ f(a_1) = g \circ f(a_2)$, and since f and g are inverses this means $a_1 = a_2$, so f is injective.

Next take $b \in B$. Then $g(b)$ is an element of A for which $f \circ g(b) = b$, so $\forall b \in B \; \exists g(b) \in A$ for which $f(g(b)) = b$, so f is surjective.

Thus f is bijective. $\square$

4. (a) A set A is equipollent to itself.

Well, $f(a_n) = a_n$ is a bijection, meaning A is equipollent to itself.

Good

(b) If A is equipollent to B, then B is equipollent to A.

If A is equipollent to B, then there is some bijective function $f: A \to B$. Since f is bijective, it has an inverse $g: B \to A$, which is also bijective. $\therefore$ B is equipollent to A.

Nice

5. The set of integers is denumerable.

Proof: The set of integers is denumerable iff there exists a bijection between the set of integers and the set of natural numbers.

The function $f(x) = \begin{cases} 2|x|, & x \leq 0 \\ 2x-1, & x > 0 \end{cases}$ is a bijection between the set of integers and the set of natural numbers. This sends all the negative integers to the even numbers and all positive integers to the odd numbers. Zero gets sent to itself. Because this bijection exists, the proposition is true. □

Great