Examlet 4 **Foundations of Advanced Math** **11/15/19**

1. Consider the relation $\sim$ on $\mathbb{R}$ defined by $x \sim y \Leftrightarrow |x - y| \leq 4$.

   (a) Find 3 elements of $\mathbb{R}$ that are related to 2.

   $-1 \sim 2$

   $1 \sim 2$

   $4 \sim 2$

   (b) Find 3 elements of $\mathbb{R}$ that are not related to 2.

   $17 \not\sim 2$

   $-3 \not\sim 2$

   $-973 \not\sim 2$

   (c) Determine whether $\sim$ is an equivalence relation.

   $x \sim x \Leftrightarrow |x - x| = 0 \leq 4$ reflexive ✓

   $x \sim y \Leftrightarrow |x-y| = |y-x| \Leftrightarrow y \sim x$ symmetric ✓

   $x \sim y$ and $y \sim z$ so $x \sim z$: Counterexample:

   $-1 \sim 2,\ 2 \sim 4,\ -1 \not\sim 4$

   transitive X

   $\boxed{\text{Not an equivalence relation}}$

Good

2. Let $S = \{a, b, c, d\}$, and let $\sim = \{(a,a), (b,b), (b,d), (c,c), (d,b), (d,d)\}$.

(a) Give the equivalence classes of $\sim$.

$[a] = \{a\}$

$[b] = \{b, d\}$

$[c] = \{c\}$

$[d] = \{b, d\}$

Great

(b) Give the partition associated with $\sim$.

$\{\{a\}, \{b, d\}, \{c\}\}$

3. Let S be a set and Π a partition of S. Let $\sim$ be a relation on S defined by $a \sim b \Leftrightarrow \exists P \in \Pi$ for which $a, b \in P$.

(a) Show $\sim$ is a reflexive relation.

Well, since Π is a partition, for any $a \in S$ we know $a \in P$ for some $P \in \Pi$, because the union of all the sets in Π is all of S. So since $a \in P$, we can also say $a, a \in P$, so $a \sim a$. $\square$

(b) Show $\sim$ is a symmetric relation.

Well, s'pose $a \sim b$, so $a, b \in P$ for some $P \in \Pi$. But then we can also say $b, a \in P$, so $b \sim a$. $\square$

(c) Show $\sim$ is a transitive relation.

Well, s'pose $a \sim b$ and $b \sim c$. Then we know $\exists P_1 \in \Pi$ for which $a, b \in P_1$, and $\exists P_2 \in \Pi$ for which $b, c \in P_2$. But then $b \in P_1 \cap P_2$, so $P_1 \cap P_2 \neq \emptyset$, so since Π was pairwise disjoint $P_1 = P_2$. Thus $a, b, c \in P_1 = P_2$, so $a \sim c$. $\square$

4. (a) Give all (unlabeled) trees with $n \leq 5$ vertices.

$n=0$:

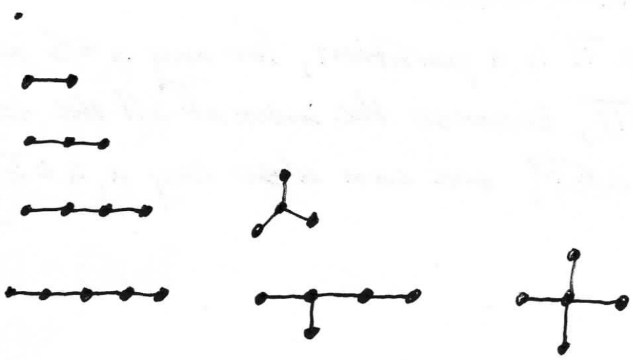
$n=1$:

$n=2$:

$n=3$:

$n=4$:

$n=5$:



(b) The number of edges in a tree with n vertices is $n-1$.

Well, it doesn't work with $n=0$ vertices, but for $n=1$ vertex we definitely have $1-1=0$ edges, and for $n=2$ vertices the only tree is , with $2-1=1$ edge.

Let's induct. S'pose it's true for any graph with k vertices, and consider some graph G with $k+1$ vertices. We know from a previous exercise that any ~~graph~~ tree with at least 2 vertices has at least 2 vertices of degree 1. Pick one of them and remove it and its edge from G. We now have a graph with k vertices, and by our inductive hypothesis it has $k-1$ edges. But G itself had one more edge than that, so $(k-1)+1$ edges, as desired. So by induction it holds for any $n \geq 2$ vertex ~~g~~ tree. $\square$

5. Call a graph **quintic** iff every vertex in that graph has degree 5. Then the number of vertices in any quintic graph must be even.

We showed in an exercise that the sum of the degrees of the vertices in a graph is always even. But then an odd number of vertices in a graph, each having degree 5, would have total degree odd times odd, which is odd, a contradiction.

Thus the number of vertices in any quintic graph must be even.

And they look really cool.