Exam 3 **Modern Algebra 1** **10/30/20**

Do questions 1 through 7 and pick three of the remaining (lettered) questions for grading (check boxes of the lettered problems you want graded or I roll dice). Each problem is worth 10 points. Show good justification for full credit. Don't panic.

1. (a) State the definition of a ring.

a set R with operation $\times$ + +. a ring must:

- be an Abelian group with respect to +
- be associative with respect to $\times$
- $a(b+c) = ab+ac$ and $(a+b)c = ac+bc$ for all $a, b, c \in R$.

Great

(b) Give an example of a ring without unity.

The set of even integers, because 1 is not in the set.

2. State the definition of a group isomorphism.

Let G be a group with operation $*$ and let H be a group with operation $\#$ and let $\emptyset$ be a mapping from G to H which is <u>one-to-one</u> and <u>onto</u> and if, for $\forall a,b \in G$ we can say

$$\underline{\emptyset(a*b) = \emptyset(a)\ \#\ \emptyset(b)}$$

then $\emptyset$ is an isomorphism, and G and H are isomorphic (can be denoted as $G \cong H$)

Good

3. State the definition of the quotient group G/N.

With group G and normal subgroup N,
we say the quotient group G/N is the set of right cosets of N in G, with the operation that if $Na, Nb \in G/N$,

$$(Na)(Nb) = N(ab).$$

Great -

$gng^{-1} \in N$

4. (a) Determine whether $\langle(1\ 2\ 3)\rangle$ is a normal subgroup of S_3.

$N = \{(1), (1\ 2\ 3), (1\ 3\ 2)\}$

$(1\ 2)(1\ 2\ 3)(1\ 2) = (1\ 3\ 2)$ ✓

$(1\ 3)(1\ 2\ 3)(1\ 2) = (1)$ ✓

$(2\ 3)(1\ 2\ 3)(2\ 3) = (1\ 3\ 2)$ ✓

$(1\ 2)(1\ 3\ 2)(1\ 2) = (1\ 2\ 3)$ ✓

$(1\ 3)(1\ 3\ 2)(1\ 3) = (1\ 2\ 3)$ ✓

$(2\ 3)(1\ 3\ 2)(2\ 3) = (1\ 2\ 3)$ ✓

$\langle(1\ 2\ 3)\rangle$ is a normal subgroup.

(b) Determine whether $\langle(1\ 2)\rangle$ is a normal subgroup of S_3.

Excellent

$N = \{(1), (1\ 2)\}$

$\underset{g}{(1\ 3)}\underset{n}{(1\ 2)}\underset{g^{-1}}{(1\ 3)} = \underline{(1)(2\ 3)} \notin N$

since $gng^{-1} \notin N$, $\langle(1\ 2)\rangle$ is not a normal subgroup of S_3.

5. Which elements of $\mathbb{Z}_{12}$ are zero divisors and why?

[2], [3], [4], [6], [8], [9], [10] are zero divisors

because for any member a of this set, there is another member b such that $a * b = 0$, $a, b \neq 0$.

Good

6. Prove that if $\theta : G \to H$ is a homomorphism, then $\ker\theta$ is a subgroup of G.

- Well, we know by earlier proof that $\Theta(e_G) = e_H$, and e_G is the identity for G, so it is also the identity of $\ker\Theta$.

- Take $a, b \in \ker\Theta$. $\Theta(a*b) = \Theta(a) \# \Theta(b) = e_H \# e_H = e_H$,

  since $\Theta(a*b) = e_H$, $a*b \in \ker\Theta$. $\ker\Theta$ is closed.

- Take $a \in \ker\Theta$. We know a^{-1} exists because G is a group.

$\Theta(aa^{-1}) = \Theta(a^{-1}a) = \Theta(e_G) = e_H$.

$\Theta(aa^{-1}) = \Theta(a) \# \Theta(a^{-1}) = e_H$

$\Theta(a) \# \Theta(a^{-1}) = e_H \# \Theta(a^{-1}) = \Theta(a^{-1}) = e_H$.

So, if $a \in \ker\Theta$, then $a^{-1} \in \ker\Theta$.

$\therefore$ $\ker\Theta$ is a subgroup of G.

Excellent

7. Prove that if $\theta : G \to H$ is a homomorphism, then ker θ is a normal subgroup of G.

Well, on the previous page, we proved it was a subgroup so all we need is the normal part.

$\forall n \in \ker\theta$ and $\forall g \in G$,

$$\theta(gng^{-1}) = \theta(g)\theta(n)\theta(g^{-1}) = \theta(g)\, e_H (g^{-1}) =$$
$$\theta(g)\theta(g^{-1}) = \theta(gg^{-1}) = \theta(e_G) = e_H$$

So because $gng^{-1} \in \ker\theta$

it is a normal subgroup.

- Excellent

~~6~~A. Show that if G is a group and N a normal subgroup of G, then the set G/N with operation as defined in question 3 forms a group.

well-defined:

we must show $N(a_1b_1) = N(a_2b_2)$. so, take $N(a_1) = N(a_2)$ and $N(b_1) = N(b_2)$. this means $a_1 = n_1 a_2$ for some $n_1 \in N$ and $b_1 = n_2 b_2$ for some $n_2 \in N$. thus, $a_1 b_1 = n_1 a_2 n_2 b_2$. Since N is a normal subgroup, we know $a_2 n_2 a_2^{-1} = n_3$ for some $n_3 \in N$, which equals $a_2 n_2 = n_3 a_2$. plugging that into the previous equation, we get $a_1 b_1 = n_1 n_3 a_2 b_2$. Since $n_1, n_3 \in N$, we have $N(a_1 b_1) = N(a_2 b_2)$ + G/N is well-defined.

ass:

take some $a, b, c \in G$ so

$(N(a)N(b))N(c) = (N(ab))N(c) = N((ab)c) = N(a(bc))$
$= (Na)N(bc) = (Na)(N(b)N(c))$ + G/N is associative.

identity:

take some $e \in G$ so

$$\left.\begin{array}{l} NeNa = Nea = Na \\ NaNe = Nae = Na \end{array}\right\} \text{ so } Ne \text{ is the identity for } G/N.$$

Excellent!

inverse:

take some $a^{-1} \in G$ so

$$\left.\begin{array}{l} Na^{-1}Na = Na^{-1}a = Ne \\ NaNa^{-1} = Naa^{-1} = Ne \end{array}\right\} \text{ so } Na^{-1} \text{ is the inverse for } G/N.$$

thus, G/N is a group. $\square$

■ B. Give an example of an integral domain which is not finite.

$\mathbb{Z} \times \mathbb{Z}$ because from a previous proof it has no zero divisors, and has unity. □.

Great

☐ C. Prove that if R is a ring and 0 the additive identity of R, then for any $a \in R$, $0 \cdot a = a \cdot 0 = 0$.

$$0a + 0a = (0+0)a = 0a = 0a + 0$$
$$-0a \qquad\qquad -0a$$

$$0a = 0$$

$$a0 + a0 = a(0+0) = a0 = a0 + 0$$
$$-a0 \qquad\qquad -a0$$

$$a0 = 0$$

Good

So, $0a = 0 = a0$

□

☒ D. Prove Lila's Theorem, that any subgroup of an Abelian group is normal.

Proof:

If G is an Abelian group, then take N as a subgroup of G.

Taken $g \in G$ and $n \in N$ so

$gng^{-1} = gg^{-1}n$ because G is abelian.

So, $gng^{-1} = gg^{-1}n = en = n \in N$.

$\therefore$ $gng^{-1} \in N$ and thus $N \trianglelefteq G$. $\square$

Great

E. Show that if G and H are isomorphic groups with G being Abelian, then H must also be Abelian.

Let G be a group with Operation $*$ and H be a group with Operation $\#$. If $X, Y \in H$ then there are elements $a, b \in G$ for which $\theta(a) = X$ and $\theta(b) = Y$. Since θ preserves the operation and G is Abelian. we can write

$X \# Y = \theta(a) \# \theta(b) = \theta(a * b) = \theta(b * a) = \theta(b) \# \theta(a) = Y \# X$ So H must be Abelion. $\square$.

Great!