Exam 2 Calc 1 10/2/2020

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the formal definition of the derivative of a function $f(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Great

2. Find an equation of the tangent line to the curve $y = x^3$ at the point (2, 8).

$y' = 3x^2$

$y'(2) = 3(2)^2 = 12 = m$

$y - y_0 = m(x - x_0)$

$y - 8 = 12(x - 2)$

$+8 \qquad +8$

$\boxed{y = 12(x-2) + 8}$

Good

3. Prove the Sum Rule for Derivatives, that if $f(x)$ and $g(x)$ are two differentiable functions then $(f+g)'(x) = f'(x) + g'(x)$.

$$(f+g)'(x)$$

$$= \lim_{h \to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) + g(x+h) - (f(x) + g(x))}{h}$$

$$= \lim_{h \to 0} \left(\underbrace{\frac{f(x+h) - f(x)}{h}} + \underbrace{\frac{g(x+h) - g(x)}{h}} \right)$$

$$= f'(x) + g'(x) \blacksquare$$

Great

4. A table of values for f, g, f', and g' is given below.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	–1	2	5	4
2	6	7	2	5
3	9	3	1	7

a) If $h(x) = f(x) \cdot g(x)$, what is $h'(2)$ and why? Product

$h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

$h'(2) = 2 \cdot 7 + 6 \cdot 5$

$= \boxed{44}$

b) If $h(x) = f(x) / g(x)$, what is $h'(3)$ and why? Quotient

$h'(x) = \dfrac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g(x^2)}$

$h'(3) = \dfrac{1 \cdot 3 - 9 \cdot 7}{3^2} = \dfrac{-60}{9}$

$\boxed{-\dfrac{60}{9}}$

c) If $h(x) = f(g(x))$, what is $h'(1)$ and why? Chain

$h'(x) = f'(g(x)) \cdot g'(x)$

$h'(1) = f'(2) \cdot 4$

$= 2 \cdot 4 = \boxed{8}$

Great

5. Differentiate

a) $f(x) = x^2 \cos x$

$2x \cdot \cos x + x^2 \cdot -\sin x$

$f'(x) \cdot g(x) + f(x) \cdot g'(x)$

Product Rule

b) $g(x) = \dfrac{\cos x}{x^2}$

$\dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

$\dfrac{(-\sin x \cdot x^2) - (\cos x)(2x)}{(x^2)^2}$

Quotient Rule

Great

c) $h(x) = (\cos x)^2$

$f'(g(x)) = f'(g(x)) \cdot g'(x)$

$2(\cos x) \cdot -\sin x$

6. State and prove the Product Rule for derivatives. Make it clear how you use any assumptions.

$$\lim_{h \to 0} \frac{(f \cdot g)(x+h) - (f \cdot g)(x)}{h}$$

$$\lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$\lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)\cdot g(x+h) + f(x)\,g(x+h) - f(x)g(x)}{h}$$

$$\lim_{h \to 0} g(x+h)\left(\frac{f(x+h) - f(x)}{h}\right) + \lim_{h \to 0} f(x)\left(\frac{g(x+h) - g(x)}{h}\right)$$

def. of derivative

def. of derivative

$$g(x) \cdot f'(x) + f(x) \cdot g'(x)$$

Great

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "*Ohmygod*. This Calculus stuff is so unfair! I swear, it's literally the least fair thing ever. There was this one function, like the cube root, right, and like, the professor was talking about it right at the end of class because somebody asked a question, right? And the question was about the tangent line when it's zero, but like, somehow the calculator said error, right? So the professor said there *is* a tangent line, but it's not wrong that the calculator said error, which is totally contradictory and unfair, but class was ending so there were, like, 200 people standing up in front of me and I have no idea what he was saying, so now I'm going to *fail!*"

   Help Bunny by explaining as clearly as possible why a calculator might get an error in connection with such a question, but the tangent line might still exist.

Dear Bunny,

Dude, its all good! Theres a totally chill reason for this! So basically there are some functions that have spots where their lines are totally vertical, right? And $\sqrt[3]{x}$ is one of those at $x=0$! So the slope of a vertical line is undefined, which is why your calculator can't figure it out! But even if a tangent line is totally vertical and has an undefined slope, it's still there!

I hope that helped!

8. Show why the derivative of the function $\tan x$ is $\dfrac{1}{\cos^2 x}$ (or $\sec^2 x$ if you prefer).

Quotient

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x \cdot \cos x - \sin x \cdot -\sin x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \leftarrow \text{trig ID!}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x \blacksquare$$

Great

9. Let $f(x) = x^{1/3}$.

a) Find $f'(x)$.

$f'(x) = \frac{1}{3}x^{-2/3}$

b) Find $f'(8)$.

$f'(8) = \frac{1}{3}(8^{-2/3})$

$= \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$ slope @ 8

c) Write an equation for the tangent line to $f(x)$ at (8,2).

$y - 2 = \frac{1}{12}(x - 8)$

d) Write the local linearization for f at $x = 8$.

$L(x) = \frac{1}{12}(x-8)+2$

Good

e) Use a local linearization to approximate $\sqrt[3]{9}$.

$\sqrt[3]{9} = x^{1/3}$, so plug in 9!

$L(x) = \frac{1}{12}(9-8)+2$

$L(x) = \frac{1}{12}(1)+2$

$\boxed{L(x) = 2.08\overline{3}}$

$9^{1/3} = 2.08$

10. a) Find the slope of the tangent line to the curve with equation $x^2 - xy + y^2 = 3$ at the point $\left(\sqrt{3}, 0\right)$.

$$2x - (1 \cdot y + x \cdot 1y') + 2y \cdot y' = 0$$

$$2x - y - xy' + 2yy' = 0$$

$$2yy' - xy' = y - 2x$$

$$y'(2y - x) = y - 2x$$

$$y' = \frac{y - 2x}{2y - x}$$

So at $(\sqrt{3}, 0)$ $\quad y' = \dfrac{(0) - 2(\sqrt{3})}{2(0) - (\sqrt{3})} = 2$

b) The point $\left(\sqrt{3}, 0\right)$ is one of the x-intercepts of the curve. Find the other x-intercept and show that the tangent line there is parallel to the one in part a.

To find x-intercepts plug in 0 for y:

$$x^2 - x(0) + (0)^2 = 3$$

$$x^2 = 3$$

$$x = \pm\sqrt{3}$$

So the other x-intercept is $(-\sqrt{3}, 0)$

And the slope there is

$$y' = \frac{(0) - 2(-\sqrt{3})}{2(0) - (-\sqrt{3})} = 2$$

So since both points get slopes of 2, the tanget lines are parallel.