Exam 1 Calc 3 9/25/2020

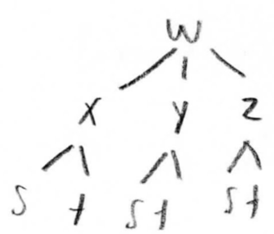
Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the formal definition of the partial derivative of a function $f(x, y)$ with respect to x.

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

Great!

2. Suppose that w is a function of x, y, and z, each of which is a function of s and t. Write the Chain Rule formula for $\dfrac{\partial w}{\partial t}$. Make very clear which derivatives are partials.



$$\underbrace{\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial t}}$$

all are partial der.
b/c there are more than
one variable

Great!

3. The function f has continuous second derivatives, and a critical point at (3, 6). Suppose $f_{xx}(3,6) = -4, f_{xy}(3,6) = -6, f_{yy}(3,6) = -9$. Classify the critical point at (3, 6).

$$D = (-4)\cdot(-9) - (-6)^2$$

$$= 36 - 36$$

$$= 0$$

Using the second derivative test, it is unclear whether the critical point (3,6) is a maximum, minimum, or saddle point since the equation ends up equaling 0.

It could be any of them or none of them

Yes!

4. Show that $\lim_{(x,y)\to(0,0)} \dfrac{x^4 - 4y^2}{x^2 + 2y^2}$ does not exist.

approach along $x=0$

$$\lim_{(0,y)\to(0,0)} \frac{(0)^4 - 4y^2}{(0)^2 + 2y^2} = \lim_{(0,y)\to(0,0)} \frac{-4}{2} = -2$$

approach along $y=0$

$$\lim_{(x,0)\to(0,0)} \frac{x^4 - 4(0)^2}{x^2 - 2(0)^2} \quad \lim_{(x,0)\to(0,0)} x^2 = 0$$

$\therefore$ The limit does not exist b/c the limit is different when approached in 2 different direction.

Excellent

5. Let $f(x, y) = \dfrac{x}{x^2 + y^2}$. Find the maximum rate of change of f at the point (1,2) and the direction in which it occurs.

$$f_x(x,y) = \frac{(x^2+y^2)(1) - (x)(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$f_x(1,2) = \frac{-(1)^2+(2)^2}{(1^2+2^2)^2} = \frac{-1+4}{(1+4)^2} = \frac{3}{25}$$

$$f_y(x,y) = \frac{(0)(x^2+y^2) - (x)(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$f_y(1,2) = \frac{-2(1)(2)}{(1^2+2^2)^2} = \frac{-4}{(1+4)^2} = \frac{-4}{25}$$

$$\sqrt{\left(\frac{3}{25}\right)^2 + \left(\frac{-4}{25}\right)^2} = \sqrt{\frac{9}{625} + \frac{16}{625}} = \sqrt{\frac{25}{625}} = \frac{5}{25} = \frac{1}{5}$$

The maximum rate of change is $\frac{1}{5}$ at the point (1,2) in the direction $\left\langle \frac{3}{25}, \frac{-4}{25} \right\rangle$

Excellent

6. Show that for any vectors $\vec{a}$ and $\vec{b}$, the vector $\vec{a} \times \vec{b}$ is perpendicular to $\vec{b}$.

Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$

We know that vectors are perpendicular when their dot product equals 0.

So...

$$\vec{a} \times \vec{b} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

Then dot product

$$\langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle \cdot \langle b_1, b_2, b_3 \rangle$$

$$= \cancel{a_2b_1b_3} - \cancel{a_3b_1b_2} + \cancel{a_3b_1b_2} - \cancel{a_1b_2b_3} + \cancel{a_1b_2b_3} - \cancel{a_2b_1b_3} = 0$$

everything cancels, equaling zero

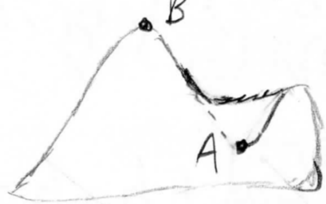
Because the dot product of these vectors is equal to zero, they are perpendicular.

Excellent!

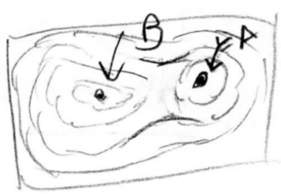
7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Crap, this Calc 3 stuff is killing me. There was this question on our practice exam about the level curves and the maxes and stuff, right? And it said if there's this level curve picture, right, and the outside curve is the lowest, then the critical point in the middle has to be a max, right? Which is totally right, 'cause that would be like a parabala thing, with the peak up on top, right?"

Explain clearly to Biff what can be said about the critical point in this situation.

Level curves that don't show the values can be really confusing, and that point could be the max, but it could also be a local minimum. Think about it like youre on a mountain and youre walking away from the peak towards a basin. That basin goes back up on the other side like this. Point A might look like it is another max like B in the second picture, but it is still a minimum. This is why it is important to label the level curves, color code them, or show it from multiple angles.



Nice



Great

8. a) Find an equation for the plane tangent to the paraboloid $z = x^2 + y^2$ at the point $(-1, 2)$.

$z_x = 2x \quad z_x(-1,2) = -2 \quad z(-1,2) = 5$

$z_y = 2y \quad z_y(-1,2) = 4$

$$z - 5 = -2(x+1) + 4(y-2)$$

or

$$z - 5 = -2x - 2 + 4y - 8$$

or

$$z = -2x + 4y - 5$$

b) Find all points on the surface $z = x^2 - y^2$ where the tangent plane is parallel to the one from part a.

$z = x^2 - y^2 \Rightarrow 0 = x^2 - y^2 - z$

$\nabla f = \langle 2x, -2y, -1 \rangle$

To be parallel to $\langle -2, 4, -1 \rangle$:

$$\langle 2x, -2y, -1 \rangle = \lambda \langle -2, 4, -1 \rangle$$

$\therefore \lambda = 1, \quad x = -1, \quad y = -2$

So $(-1, -2, -3)$

$-15\cos\left(\frac{\pi x}{12} - \frac{\pi}{4}\right) + \frac{y}{50} + 80$

9. Let $T(x, y) = -15\cos(\pi x/12 - \pi/4) + y/50 + 80$. Find the directional derivative of T in the direction of the vector $\langle 1,60\rangle$. at $(10, 0)$

$\bar{v} = \langle 1, 60\rangle \quad |\bar{v}| = \sqrt{1^2 + 60^2} = \sqrt{3601}$

$\bar{u} = \left\langle \frac{1}{\sqrt{3601}}, \frac{60}{\sqrt{3601}} \right\rangle = \left\langle \frac{\sqrt{3601}}{3601}, \frac{60\sqrt{3601}}{3601} \right\rangle$

$D_u T(10,0) = \nabla T(10,0) \cdot \bar{u}$

$\nabla T(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$

$f_x(x,y) = +15 \sin\left(\frac{\pi x}{12} - \frac{\pi}{4}\right) \cdot \frac{\pi}{12} = \frac{5\pi}{4} \sin\left(\frac{\pi x}{12} - \frac{\pi}{4}\right)$

$f_y(x,y) = \frac{1}{50}$

$f_x(10, 0) = \frac{5\pi}{4} \sin\left(\frac{5\pi}{6} - \frac{\pi}{4}\right) = \frac{5\pi}{4} \sin\left(\frac{7\pi}{12}\right) = \frac{\sqrt{6} + \sqrt{2}}{4}$

$f_y(10,0) = \frac{1}{50}$

$\nabla T(10,0) = \left\langle \frac{5\pi}{4} \sin\left(\frac{7\pi}{12}\right), \frac{1}{50} \right\rangle$

$D_u T = \left\langle \frac{5\pi}{4} \sin\left(\frac{7\pi}{12}\right), \frac{1}{50} \right\rangle \cdot \left\langle \frac{\sqrt{3601}}{3601}, \frac{60\sqrt{3601}}{3601} \right\rangle$

$= \frac{5\pi\sqrt{3601}}{14404} \sin\left(\frac{7\pi}{12}\right) + \frac{6\sqrt{3601}}{18005}$

$= \frac{5\pi\sqrt{3601}}{14404} \cdot \frac{\sqrt{6}+\sqrt{2}}{4} + \frac{6\sqrt{3601}}{18005}$

Good

$= \frac{5\pi\left(\sqrt{21606} + \sqrt{7202}\right)}{57616} + \frac{6\sqrt{3601}}{18005} \approx 0.0832$

10. Find all critical points of $f(x,y) = xy(1 - x - y)$ and classify them as maxima, minima, or saddle points.

$f_x(x,y) = y(1-x-y) + (xy)(-1)$

$y - xy - y^2 - xy$

$-y^2 + y - 2xy$

$y(-y+1-2x) = 0 \qquad y = 0$

$f_{xx} = -2y$

$f_{yy} = -2x$

$f_{xy} = -2y + 1 - 2x$

$f_{yx} = -2x + 1 - 2y$

$f_y(x,y) = x(1-x-y) + (xy)(-1)$

$x - x^2 - xy - xy$

$-x^2 + x - 2xy$

$x(-x+1-2y) = 0 \qquad x = 0$

10

Excellent!

When $x=0$

$y(-y+1-2(0)) = 0$

$y(-y+1) = 0$

$y = 0$

$y = 1$

$(0,0)$
$(0,1)$

when $y=0$

$x(-x+1-2(0)) = 0$

$x(-x+1) = 0$

$x = 0$

$x = 1$

$(0,0)$
$(1,0)$

$-y+1-2x = 0$

$1-2x = y$

$-x+1-2(1-2x) = 0$

$-x+1-2+4x = 0$

$-1+3x = 0$

$3x = 1$

$x = \frac{1}{3}$

When $x = \frac{1}{3}$

$-y+1-2\frac{1}{3} = 0$

$-y+1-\frac{2}{3} = 0$

$-y+\frac{1}{3} = 0$

$\frac{1}{3} = y$

$(\frac{1}{3}, \frac{1}{3})$

Critical Points

$(0,0)$

$(1,0)$

$(0,1)$

$(\frac{1}{3}, \frac{1}{3})$

$(f_{xx})(f_{yy}) - (f_{xy})^2$

$D\ (0,0) = (0)(0) - (0+1-0)^2 = -1$ $\boxed{\text{SADDLE}}$

$(0,1) = (-2)(0) - (-2+1)^2 = 0 - 1 = -1$ $\boxed{\text{SADDLE}}$

$(1,0) = (0)(-2) - (0+1-2)^2 = -1$ $\boxed{\text{SADDLE}}$

$(\frac{1}{3}, \frac{1}{3}) = (\frac{-2}{3})(\frac{-2}{3}) - (\frac{-2}{3} + 1 - \frac{2}{3})^2 =$

$(\frac{4}{9}) - \frac{1}{9} = \boxed{\frac{1}{3}}$

$f_{xx} = -2(\frac{1}{3}) = \text{negative}$

$\boxed{\text{max}}$