Exam 2 **Real Analysis 1** **11/6/20**

Each problem is worth 10 points. Show adequate justification for full credit. Don't panic.

1. State the definition of the derivative of a function $f(x)$ at $x = a$.

Let $f: D \to \mathbb{R}$ with $D \subseteq \mathbb{R}$. Then the derivative of $f(x)$ at $x = a$ is $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ provided that the limit exists, meaning a must be an accumulation point of f, and that it is finite.

Good

2. a) State the definition of a set E being closed.

A set E is closed iff all the accumulation points of E are elements of E.

b) State the definition of a set E being open.

A set E is open iff $\forall a \in E, \exists \delta > 0 \ni (a-\delta, a+\delta) \subseteq E$

Great

Intermediate

3. State the ~~Indermediate~~ Value Theorem.

IVT: If f is a continuous function on $[a, b]$

and $k \in \mathbb{R}$ is between $f(a)$ and $f(b)$, then

$\exists c \in (a, b) \ni f(c) = k$.

Good

4. a) State the definition of a compact set.

A set S is compact iff every open cover of S has a finite sub cover.

b) State the Heine-Borel Theorem.

In $\mathbb{R}$, a set is compact iff it's both closed and bounded.

Excellent

c) Give an example of an open cover for $(0, 2020)$ that has no finite subcover.

$$A = \bigcup_{n \in \mathbb{N}} \left(\frac{1}{n}, 2020\right)$$

5. Prove that if f is differentiable at a then f is continuous at a.

Proof: Since f is differentiable at a, a is an accumilation point of the domain of f (#1 note).

Since a is an a.p., f is continuous iff $\lim_{x\to a} f(x) = f(a)$ or $\lim_{x\to a} f(x) - f(a) = 0$.

Now, $\lim_{x\to a} f(x) - f(a) = \lim_{x\to a} (f(x) - f(a))$ since $f(a)$ is a constant. For this limit $x \neq a$ so $x - a \neq 0$, and thus

$$\lim_{x\to a} (f(x) - f(a)) = \lim_{x\to a} \left(\frac{f(x) - f(a)}{x-a} (x-a) \right)$$

$$= \lim_{x\to a} \left(\frac{f(x) - f(a)}{x-a} \right) \cdot \lim_{x\to a} (x-a)$$

Since both limits exist, The first limit exists because $f'(a)$ exists and the second limit exists because $x-a$ is a polynomial. $f'(a)$ exists since f is differentiable at a

$$= f'(a) \cdot (a-a) = f'(a) \cdot 0 = 0$$

Excellent

$\therefore f$ is continuous at a

6. State and prove Fermat's Theorem.

Fermat's Theorem: Let $f: D \to \mathbb{R}$, with $D \subseteq \mathbb{R}$. If f has a relative extrema at $x = c \in (a,b) \subseteq D$, and f is differentiable at c, then $f'(c) = 0$.

Proof: We'll prove just the max. case but the min. follows. Well, if there's a relative maximum at $x = c$, then by def^{n}, $\exists \delta > 0 \ni |x-c| < \delta \wedge x \in D \Rightarrow f(x) \leq f(c)$ $\Rightarrow f(x) - f(c) \leq 0$. Now for $x > c \wedge x \in (c-\delta, c+\delta) \cap D$,

$$x - c > 0 \Rightarrow \frac{f(x)-f(c)}{x-c} \leq 0 \Rightarrow \lim_{x \to c^+} \frac{f(x)-f(c)}{x-c} \leq 0.$$

For $x < c \wedge x \in (c-\delta, c+\delta) \cap D$,

$$x - c < 0 \Rightarrow \frac{f(x)-f(x)}{x-c} \geq 0 \Rightarrow \lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} \geq 0.$$

But we know that $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists and is finite, by the def^{n} of differeniability, so the RHS & LHS limits must be equal. This is the case only when

$$\lim_{x \to c^+} \frac{f(x)-f(c)}{x-c} = \lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} = 0.$$

Excellent

$\therefore \lim_{x \to c} \frac{f(x)-f(c)}{x-c} = f'(c) = 0$, as desired.

7. State and prove Rolle's Theorem.

theorem: Let f be a function for which

- f is continuous on $[a,b]$
- f is differentiable on (a,b)
- $f(a) = f(b)$

then $\exists c \in (a,b) \ni f'(c) = 0$.

proof: Well, consider the cases...

- Well, if f is a constant function then there is guaranteed to be some $c \in (a,b)$.
- Well, if $\exists x \in (a,b)$ where $f(x) > f(a)$ then by EVT f has some $c \in [a,b]$ where $f(c)$ is the maximum. But $c \neq a$ and $c \neq b$ so $c \in (a,b)$ and by Fermat's Theorem $f'(c) = 0$.
- Well if $\exists x \in (a,b)$ where $f(x) < f(a)$. Then it follows the same steps as case 2 but with a minumum rather than the maximum.

Since all cases are true, the theorem itself is true. $\square$

Excellent!

☑B. Prove that the product of continuous functions is continuous.

Let $f, g : D \to \mathbb{R}$, with $D \subseteq \mathbb{R}$ both be continuous at $a \in D$. Let $\varepsilon > 0$ be given. Then $\exists\, \delta_1 > 0 \ni |x-a| < \delta_1 \wedge x \in D \Rightarrow |g(x) - g(a)| < \frac{\varepsilon}{2|f(a)|+1}$. Now because f is continuous at a, by previous proof we know that $\exists\, \delta_2 > 0 \ni |x-a| < \delta_2 \wedge x \in D \Rightarrow |g(x)| < K$, $\exists K \neq 0$. Then $\exists\, \delta_3 > 0 \ni |x-a| < \delta_3 \wedge x \in D \Rightarrow |f(x) - f(a)| < \frac{\varepsilon}{2K}$. Let $\delta^* = \min\{\delta_1, \delta_2, \delta_3\}$

Then for $|x-a| < \delta^* \wedge x \in D$,

$$
\begin{aligned}
|(fg)(x) - (fg)(a)| &= |f(x)g(x) - f(a)g(a)| \\
&= |f(x)g(x) - g(x)f(a) + g(x)f(a) - f(a)g(a)| \\
&= |(g(x))(f(x) - f(a)) + f(a)(g(x) - g(a))| \\
&\leq |g(x)||f(x) - f(a)| + |f(a)||g(x) - g(a)| \\
&< K \cdot \frac{\varepsilon}{2K} + |f(a)| \cdot \frac{\varepsilon}{2|f(a)|+1} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

$\therefore$ fg is continuous at $a \in D$, and because a was arbitrary, fg is continuous.

Excellent!

☑ C. State and prove the Boundedness Theorem.

Boundedness Theorem: If f is a continuous function on $[a,b]$, then f is bounded on $[a,b]$.

Proof: Well suppose f is not bounded on $[a,b]$, so then there exists a sequence $\{x_n\}$ in $[a,b]$ such that $|f(x_n)| > n$, $\forall n \in \mathbb{N}$. But because $\{x_n\}$ is bounded, by the BWT for sequences, there exists a subsequence $\{x_{n_k}\}$ that converges to some c. Well c must be an accumulation point of $[a,b]$, and because $[a,b]$ is closed, then $c \in [a,b]$ by def^n. But this means f is continuous at c, so by a previous proof (that someone has done...)

$$\lim_{k\to\infty} f(x_{n_k}) = f\left(\lim_{k\to\infty} x_{n_k}\right) = f(c).$$

But this contradicts our early statement as $|f(x_{n_k})| > n_k$, $\forall k \in \mathbb{N}$. Therefore f must be bounded on $[a,b]$.

Great

☒ E. Give an example of a function that is defined for all real inputs, but continuous nowhere.

$$f(x) = \begin{cases} 0 & \text{for } x \in \mathbb{Q} \\ 1 & \text{for } x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

Great

$\forall a \in \mathbb{R}$ and $\forall\, 0 < \varepsilon < 1$, there does not exist a $\delta > 0$ such that $|x-a| < \delta$ and $x \in D \Rightarrow |f(x)-f(a)| < \varepsilon$,

But f is defined on the Reals.