Exam 2 Calc 1 10/1/2021

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the formal definition of the derivative of a function $f(x)$.

If $f(x)$ is a differentialable function then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Good

$(x+h)(x+h)$

2. Use the definition of the derivative to show that if $f(x) = x^2$, then $f'(x) = 2x$.

$f(x) = x^2$

$f'(x) = 2x$

$$f'(x^2) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + xh + xh + h^2 \cancel{- x^2}}{h}$$

$$= \lim_{h \to 0} \frac{(2xh + h^2)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x + h)}{\cancel{h}}$$

$$= \lim_{h \to 0} \frac{2x + h}{1} = \lim_{h \to 0} 2x + h = 2x$$

Great!

3. Let $f(x) = \sqrt{x}$. Use the definition of the derivative to find $f'(x)$.

$f(x) = \sqrt{x}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x+0} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

Excellent!

4. Prove the Constant Multiple Rule for Derivatives, that if $f(x)$ is a differentiable function and c is a constant, then $(cf)'(x) = cf'(x)$. If $f(x)$ is a differentiable function

$$(cf)'(x) = \lim_{h\to 0} \frac{cf(x+h) - c(f(x))}{h}$$

$$= \lim_{h\to 0}\left(c\left(\frac{f(x+h)-f(x)}{h}\right)\right)$$

deffinition of derevitive

$$= \lim_{h\to 0} c \cdot \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

No h value so $\lim_{h\to 0}$ is just c

$$= c \cdot f'(x)$$

$$(cf)'(x) = cf'(x)$$

Excellent

5. [WeBWorK/Stewart 5th] In this exercise, we estimate the rate at which the total personal income is rising in the Richmond-Petersburg, Virginia, metropolitan area. In 1999, the population in this area was 961400, and the population was increasing at roughly 9200 people per year. The average annual income was 30593 dollars per capita, and this average was increasing at about 1400 dollars per year.

Use the these figures to estimate the rate at which the total personal income was rising in the Richmond-Petersburg area in 1999.

Product rule $(fog)(x) = f'(x)g(x) + f(x)g'(x)$

$f(1999) = 961400$ $(fog)(x) = 9200(30593) + 961400(1400)$

$f'(1999) = 9200$ $= 2.814556 \times 10^{8} + 1.34596 \times 10^{9}$

$g(1999) = 30593$ $\boxed{(fog)(x) = 1.6274156 \times 10^{9}}$

$g'(1999) = 1400$

Excellent

6. State and prove the Quotient Rule for derivatives. Make it clear how you use any assumptions.

$$\left(\frac{f}{g}\right)'(x) = \lim_{h\to 0} \frac{\left(\frac{f}{g}\right)(x+h) - \left(\frac{f}{g}\right)(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \lim_{h\to 0} \frac{g(x)\,f(x+h) - f(x)\,g(x+h)}{h\,g(x+h)\,g(x)}$$

$$\lim_{h\to 0} \frac{g(x)\,f(x+h) - f(x)\,g(x+h) + f(x)\,g(x) - f(x)\,g(x)}{h\,g(x+h)\,g(x)}$$

$$= \lim_{h\to 0} \frac{g(x)\,f(x+h) - f(x)\,g(x) - f(x)\,g(x+h) + f(x)\,g(x)}{h\,g(x+h)\,g(x)}$$

$$= \lim_{h\to 0}\left[\frac{1}{g(x+h)g(x)} \cdot \left(g(x) \cdot \frac{f(x+h)-f(x)}{h} - f(x) \cdot \frac{g(x+h)-g(x)}{h}\right)\right]$$

$$= \lim_{h\to 0} \frac{g(x)f'(x) - f(x)g'(x)}{g(x+h)\,g(x)}$$

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Provided f and g are differentiable functions and $g(x) \neq 0$

Yes.

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "*Ohmygod.* This Calculus stuff is so unfair! I swear, it's literally the least fair thing ever. There was this one function, like the cube root, right, and like, the professor was talking about it right at the end of class because somebody asked a question, right? And the question was about the tangent line when it's zero, but like, somehow the calculator said error, right? So the professor said there *is* a tangent line, but it's not wrong that the calculator said error, which is totally contradictory and unfair, but class was ending so there were, like, 200 people standing up in front of me and I have no idea what he was saying, so now I'm going to *fail!*"

   Help Bunny by explaining as clearly as possible why a calculator might get an error in connection with such a question, but the tangent line might still exist.

Yo, take a look at the graph of $f(x)=\sqrt[3]{x}$. At the point (0,0) if you draw in a tangent line its going to be almost vertical. Even though we can put a tangent line there, the calculator cant give us a slope because its vertical. The real slope is approaching ∞ so the calculator cant show it so it says undefined

Good.

8. a) Find the linearization $L(x)$ of the function $f(x) = \sqrt{x}$ at $a = 4$.

$f'(x) = (\sqrt{x})'$

$f'(x) = \dfrac{1}{2\sqrt{x}}$

Then: $f'(4) = \dfrac{1}{2\sqrt{4}}$ and $f(4) = \sqrt{4} = 2$

$f'(4) = \dfrac{1}{4}$

Therefore: $L(x) = f'(a)(x - a) + f(a)$

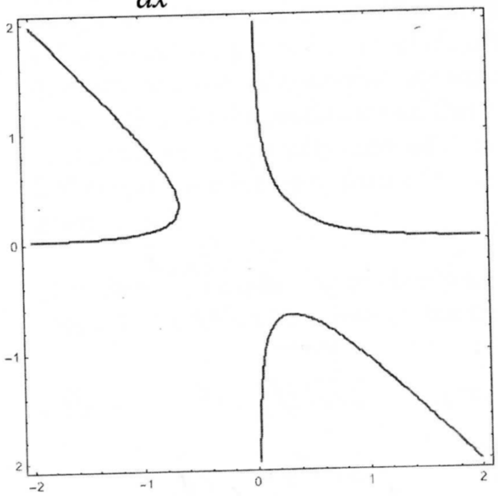
$L(4) = \dfrac{1}{4}(x - 4) + 2$

b) Use your linearization from part a to approximate $\sqrt{4.1}$

Excellent!

$\sqrt{4.1} \approx \dfrac{1}{4}(4.1 - 4) + 2 = 2.025$

9. a) Find $\frac{dy}{dx}$ for the curve with equation $x^2y + xy^2 = 2/27$.



Imp Diff

$x^2y' + 2xy + x2yy' + y^2 = 0$

Solve for y'

$x^2y' + 2yy'x = -2xy - y^2$

$y'(x^2 + 2yx) = -2xy - y^2$

$$\boxed{y' = \frac{-2xy - y^2}{x^2 + 2yx}}$$

b) Find the equation of the tangent line to the curve with equation $x^2y + xy^2 = 2/27$ at the point (1/3,1/3).

$$y'(1/3, 1/3) = \frac{-2(1/3)\cdot(1/3) - (1/3)^2}{(1/3)^2 + 2(1/3)(1/3)} = \frac{-\frac{2}{9} - \frac{1}{9}}{\frac{1}{9} + \frac{2}{9}} = \frac{-\frac{3}{9}}{\frac{3}{9}} = \boxed{-1}$$

$y - 1/3 = -1(x - 1/3)$

$y - 1/3 = -x + 1/3$

$+1/3 \qquad +1/3$

$$\boxed{y = -x + 2/3}$$

Excellent

10. Suppose $L(x)$ is a function for which $L'(x) = 1/x$ (for values of x that aren't 0).

a) Let $g(x) = L(\cos x)$. What is $g'(x)$?

$$g'(x) = \frac{1}{\cos x} \cdot (\cos x)' \qquad \text{Chain Rule!}$$

$$= \frac{1}{\cos x} \cdot -\sin x$$

$$= -\frac{\sin x}{\cos x}$$

$$= -\tan x$$

b) Let $h(x) = L\left(x + \sqrt{1+x^2}\right)$. What is $h'(x)$?

$$h'(x) = \frac{1}{x+\sqrt{1+x^2}} \cdot \left(x + \sqrt{1+x^2}\right)' \qquad \text{Chain Rule!}$$

$$= \frac{1}{x+\sqrt{1+x^2}} \cdot \left(1 + \frac{1}{2}(1+x^2)^{-1/2} \cdot 2x\right)$$

$$= \frac{1}{x+\sqrt{1+x^2}} \cdot \left(1 + \frac{x}{\sqrt{1+x^2}}\right)$$

$$= \frac{1}{x+\sqrt{1+x^2}} \cdot \left(\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} + \frac{x}{\sqrt{1+x^2}}\right)$$

$$= \frac{1}{x+\sqrt{1+x^2}} \cdot \frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}$$

$$= \frac{1}{\sqrt{1+x^2}}$$