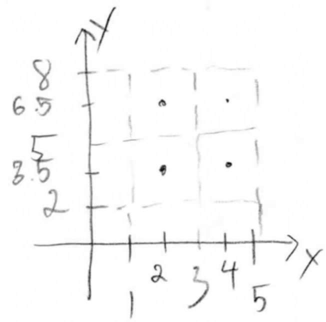
Exam 2 Calc 3 10/22/2021

Each problem is worth 10 points. For full credit provide complete justification for your answers. All integrals should be set up in terms of a single coordinate system, i.e., if you use cylindrical your integral should involve no x or y, etc.

1. Write a double Riemann sum for $\iint_R f\,dA$, where $R = \{(x, y) : 1 \le x \le 5, 2 \le y \le 8\}$ using midpoints with $n = m = 2$ subdivisions.

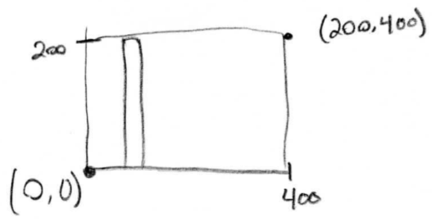


$\Delta Y = 2 \quad \Delta Y = 3$

$6 \cdot \left(f(2, 3.5) + f(4, 3.5) + f(2, 6.5) + f(4, 6.5) \right)$

2. Kansas is close enough to a rectangle 400 miles from west to east and 200 miles from south to north. Suppose that in some year the wheat harvest in Kansas is given by $w(x, y) = 4000 + 10x + 15y$ bushels per square mile, where we consider Kansas to be located on a standard coordinate system with the southwest corner positioned at (0,0). Set up a double integral for the total wheat harvest in Kansas for this year.

Top View

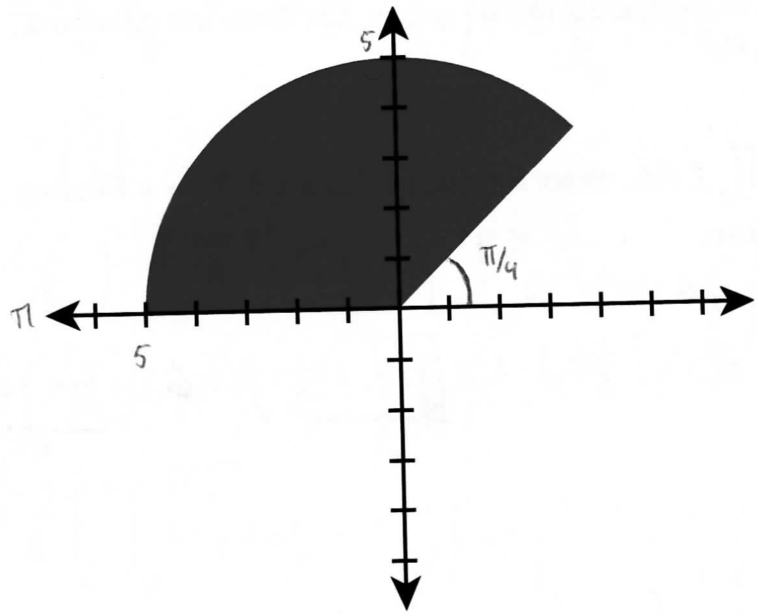


$\boxed{\int_0^{400} \int_0^{200} 4000 + 10x + 15y \; dy\,dx}$

Good

W

3. Set up an iterated integral for the total mass of a plate shaped like the region shown below, with density $\rho(x, y) = 5$.



$$\int_{\pi/4}^{\pi} \int_{0}^{5} \rho(x,y)\, dA = \int_{\pi/4}^{\pi} \int_{0}^{5} 5r\, dr\, d\theta$$

Good

$\to \iiint_E 1\, dV$

4. Set up an iterated integral for the volume of the region under $z = 36 - x^2 - y^2$ but above the xy-plane.

$$\boxed{\int_0^{2\pi} \int_0^{6} \int_0^{36-r^2} 1 \cdot r\, dz\, dr\, d\theta}$$

Cylindrical

$z = 36 - r^2$

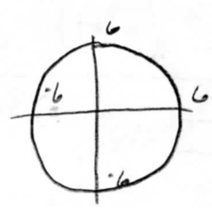
Good

$z = 0 = 36 - x^2 - y^2$

$x^2 + y^2 = 36$

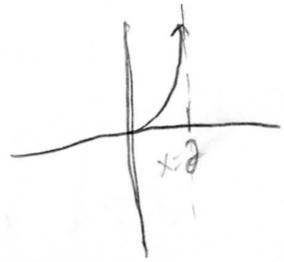
$r = 6$

Top View



L@TTV ☺

5. Evaluate $\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+1}\, dx\,dy$.



$x=\sqrt{y}$

$y\Big|_0^{x^2}$ $\quad x\Big|_0^2$

So rearrange limits of Integration to get

$$\int_0^2 \int_0^{x^2} \sqrt{x^3+1}\; dy\,dx$$

$$\int_0^2 \sqrt{x^3+1}\; y\Big]_0^{x^2}\, dx$$

$$\int_0^2 x^2\sqrt{x^3+1}\; dx$$

Then u-sub

$u = x^3+1 \longrightarrow u\Big|_1^9$ limits

$du = 3x^2\,dx$; we have x^2dx

$\frac{1}{3}du = x^2\,dx$

$$\frac{1}{3}\int u^{\frac{1}{2}}\,du$$

$$\frac{1}{3}\left(\frac{2}{3}u^{\frac{3}{2}}\Big|_1^9\right)$$

$$\frac{2}{9}(27-1)$$

$$=\boxed{\frac{2}{9}\cdot 26}$$

Excellent

6. Show that the Jacobian for the conversion from rectangular to polar coordinates is what it is.

$x = r \cdot \cos\theta$
$y = r \cdot \sin\theta$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r \cdot \cos^2\theta - -r \cdot \sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta)$$

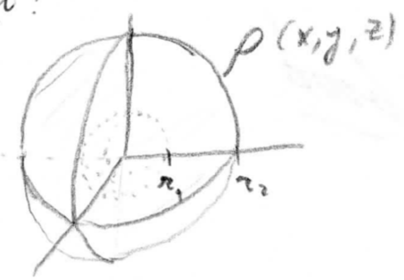
$$= r(1) = \boxed{r}$$

Great

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "OMG! Calc 3 is just so much! It's like, there's a,lways another thing, right? So like why would I ever*everever* use spherical coordinates for anything? I mean you can totally do a sphere in cylindrical coordinates, right? Just stop already!"

Explain clearly to Bunny when there might be situations in which spherical coordinates should be appreciated.

There could be a situation in which cylindrical coordinate will be really cumbersome, lets say that they want the density of a spherical crust:



In order to use cylindrical coordinate, you'll need at least three integrals, one for the outside of the ecuator of the interior sphere, another for the "stuff" in top of it and another for the stuff bellow it.

On the other hand, one simple spherical integral will do the work:

$$\int_0^{2\pi}\int_0^{\pi}\int_{r_1}^{r_2} \rho(r,\phi,\theta)\, r^2 \sin\phi \; dr\, d\phi\, d\theta$$

Excellent.

8. Evaluate $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} 6\,dz\,dy\,dx$.

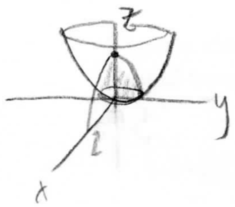
First-octant portion of a sphere with $r=2$,

so

$$V = \frac{1}{8} \cdot 6 \cdot \frac{4}{3}\pi (2)^3$$

$$= 8\pi$$

9. Set up an iterated integral to integrate $f(x,y,z) = 12xz$ over the region in the first octant above the parabolic cylinder $z = y^2$ and below the paraboloid $z = 8 - 2x^2 - y^2$.



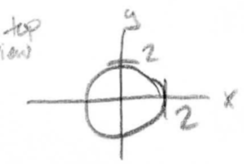
$$y^2 = 8 - 2x^2 - y^2$$

$$2y^2 + 2x^2 = 8$$

$$x^2 + y^2 = 4$$

$$y = \sqrt{4 - x^2}$$

top view



$$z\Big|_{y^2}^{8-2x^2-y^2}$$

$$y\Big|_{0}^{\sqrt{4-x^2}}$$

$$x\Big|_{0}^{2}$$

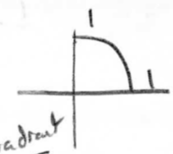
$$\boxed{\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{y^2}^{8-2x^2-y^2} 12xz \, dz\, dy\, dx}$$

Great

10. A lamina occupies the part of a disk $x^2 + y^2 \leq 1$ in the first quadrant. Set up iterated integrals to find the center of mass if the density at any point is proportional to its distance from the x-axis.

Proportional to distance from x- axis $\rightarrow$ $\underline{Ky = \rho(x,y)}$

$\underline{\text{Top}}$
$\underline{\text{View}}$



$\underline{\text{First quadrant}}$

$\underline{\theta = 0 \rightarrow \frac{\pi}{2}}$

$\underline{x = r\cos\theta}$

$\underline{y = r\sin\theta}$

$$\boxed{\begin{aligned} \bar{x} &= \frac{\int_0^{\frac{\pi}{2}} \int_0^1 k(r\sin\theta)(r\cos\theta) \cdot r\,dr\,d\theta}{\int_0^{\frac{\pi}{2}} \int_0^1 k(r\sin\theta) \cdot r\,dr\,d\theta} \\ \bar{y} &= \frac{\int_0^{\frac{\pi}{2}} \int_0^1 k(r\sin\theta)^2 r\,dr\,d\theta}{\int_0^{\frac{\pi}{2}} \int_0^1 k(r\sin\theta) \cdot r\,dr\,d\theta} \end{aligned}}$$

$\underline{\text{Excellent!}}$