Exam 2a **Calc 1** **10/7/22**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. State the formal definition of the derivative of a function $f(x)$.

As long as $f(x)$ is a differentiable function then.....

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Good

2. Use the definition of the derivative to find the derivative of $f(x) = \sqrt{x}$.

$$f(x) = \sqrt{x}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + \cancel{h} - \cancel{x}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

Good

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

3. Use the following table of values for $f(x)$ and $g(x)$ to find values for the following:

x	1	2	3	4	5	6
$f(x)$	6	3	5	1	4	2
$g(x)$	2	1	6	3	5	4
$f'(x)$	2	4	1	5	7	8
$g'(x)$	5	9	7	11	2	12

(a) If $h(x) = f(x) \cdot g(x)$, what is $h'(1)$ and why?

$h(x) = f(x) \cdot g(x)$

$h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x) = 2 \cdot 2 + 6 \cdot 5 = \boxed{34}$

because of the product rule

(b) If $h(x) = \frac{f(x)}{g(x)}$, what is $h'(6)$ and why?

$h'(x) = \dfrac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$

$h'(x) = \dfrac{8 \cdot 4 - 2 \cdot 12}{4^2} = \dfrac{8}{16} = \boxed{\frac{1}{2}}$ because of the quotient rule

(c) If $h(x) = f(g(x))$, what is $h'(4)$ and why?

$h'(x) = f'(g(x)) \cdot g'(x)$

$h'(x) = f'(3) \cdot 11$

$h'(x) = 1 \cdot 11 = \boxed{11}$ because of the chain rule

(d) If $h(x) = 5 \cdot g(x) + 20$, what is $h'(3)$ and why?

$h'(x) = 5 \cdot g'(x)$

$h'(x) = 5 \cdot 7 = \boxed{35}$ because the derivative of a constant is zero, + the 5 acts as a coefficient of $g(x)$

(e) If $h(x) = f(x^2)$, what is $h'(2)$ and why?

$h'(x) = f'(x^2) \cdot 2x$

$h'(2) = f'(2^2) \cdot 2 \cdot 2 = f'(4) \cdot 4 = 5 \cdot 4 = 20$

4. Prove that $(f+g)'(x) = f'(x) + g'(x)$ for any differentiable functions f and g.

$$(f+g)'(x) = \lim_{h\to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - (f(x) + g(x))}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) + g'(x)$$

Good.

5. [Stewart] A cylindrical tank with radius 5 m is being filled with water at a rate of 3 m^3/ min. How fast is the height of the water increasing?

$V = \pi r^2 h$

$\frac{dV}{dt} = \left(2\pi r \frac{dr}{dt} \cdot h\right) + \left(\pi r^2 \frac{dh}{dt}\right)$

$r = 5$ change in Volume = 3

$3 = \left(2\pi(5) \cdot 0\right) + \left(\pi(5)^2 \cdot \frac{dh}{dt}\right)$

$3 = 25\pi \frac{dh}{dt}$

Excellent

$\frac{dh}{dt} = \frac{3}{25\pi}$ m/min

6. State and prove the Product Rule.

assuming $f(x)$ and $g(x)$ are differentiable functions $(f \cdot g)'(x) = f \cdot g' + g \cdot f'$

$$= \lim_{h \to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)\,g(x+h) - f(x+h)\,g(x) + f(x+h)\,g(x) - f(x)\,g(x)}{h}$$

$$= \lim_{h \to 0} \left[f(x+h) \frac{g(x+h) - g(x)}{h} + g(x) \frac{f(x+h) - f(x)}{h} \right]$$

as long as $f(x)$ and $g(x)$ are differentiable

Okay.

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "OMG! Why does calculus have to be so confusing, like, they're literally trying to kill us? The professor definitely said that the derivative of 5 is zero on Monday, but then totally said that the derivative of 5 is 5 on Wednesday. Is it really different on different days?"

Help Bunny by explaining as clearly as you can what's going on.

the derivative of a constant when it is being added is zero, however, if a costant is being multiplied by something, the derivative of the constant is the same as the constant.

Excellent!

8. Use a local linearization for $f(x) = x^4$ at $x = 2$ to approximate $(2.001)^4$.

$f(x) = x^4$ $\quad$ $f(2) = 16$ $\quad$ $y - 16 = 32(x - 2)$

$\qquad\qquad\qquad\qquad\qquad\qquad +16 \qquad\qquad +16$

$f'(x) = 4x^3$ $\quad$ $f'(2) = 32$

$L(x) = 32(2.001 - 2) + 16$

$L(x) = 16.032$

Good

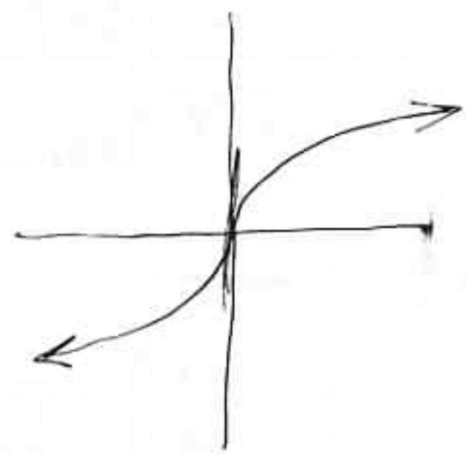
9. Find an equation for the line tangent to $f(x) = \sqrt[3]{x}$ at the point (0,0).

$f'(x) = \frac{1}{3} x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$

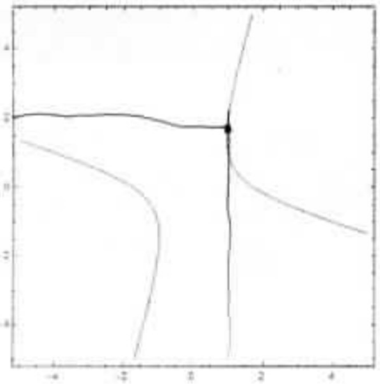
$f'(0) = \frac{1}{3\sqrt[3]{0^2}} = \infty ?!$

That's weird! But look at the graph!
∞ is the slope of a vertical line, so...

$x = 0$



10. Find an equation of the line tangent to $x^2 + 3xy - y^2 = 3$ at the point $(1, 2)$.



$2x + [3(1 \cdot y) + (x \cdot y')] - 2y \cdot y' = 0$

$2x + 3y + 3xy' - 2y \cdot y' = 0$

$3xy' - 2y \cdot y' = -2x - 3y$

Great!

$y'(3x - 2y) = -2x - 3y$

$y' = \dfrac{-2x - 3y}{3x - 2y}$

$\dfrac{-2(1) - 3(2)}{3(1) - 2(2)} = \dfrac{-2-6}{3-4}$

$= \dfrac{-8}{-1} = 8$

$y - y_1 = m(x - x_1)$

$y - 2 = 8(x - 1)$