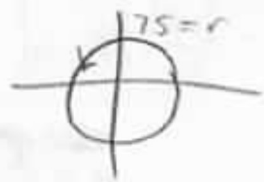
Exam 3 **Calc 3** **11/29/22**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. Give a parametrization and bounds for t to produce a counterclockwise circle with radius 75 centered at the origin.

$x(t) = 75\cos(t)$
$y(t) = 75\sin(t)$
$0 \le t \le 2\pi$



Good

2. Is the vector field $\mathbf{F}(x, y) = \langle xy^2, x + y^2\rangle$ conservative? How can you be sure?

$f_x = xy^2 \qquad f_{xy} = 2xy$
$f_y = x+y^2 \qquad f_{yx} = 2y$

This vector field is not conservative as there is no potential function $f(x,y)$ where its gradient is $\langle xy^2, x+y^2\rangle$. According to Clairaut's Theorem, the mixed partials f_{xy} and f_{yx} would need to be equivalent for a potential function to exist (conservative), however (as shown above), $f_{xy} = 2xy$ and $f_{yx} = 2y$ which are not equivalent.

Excellent!

3. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = 5\mathbf{i} + 3x\mathbf{j}$ and C is a path composed of a line segment from $(0, 0)$ to $(2, 0)$ followed by a line segment from $(2, 0)$ to $(0, 1)$ and then a line segment from $(0, 1)$ back to $(0, 0)$.

Closed Path

Use Green's Thm.

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D 3 \, dA$$

$$= 3 \text{ (area of triangle)} = 3\left(\tfrac{1}{2}\right)(2)(1) = 3$$

Excellent!

4. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = 2xy^3\mathbf{i} + 3x^2y^2\mathbf{j}$ and C is a line segment from $(1, 1)$ to $(2, 3)$.

$f(x,y) = x^2y^3$ is a potential function, so use F.T.L.I.!

$$\int \vec{F} \cdot d\vec{r} = f(2,3) - f(1,1)$$

$$= 2^2 \cdot 3^3 - 1^2 \cdot 1^3$$

$$= 4 \cdot 27 - 1 \cdot 1$$

$$= 108 - 1$$

$$= 107$$

5. Let $\mathbf{F}$ be the vector field $\mathbf{F}(x, y, z) = 2x\mathbf{i} + 5y\mathbf{j} + 3z\mathbf{k}$. Let S be the sphere with radius 3, centered at the origin and oriented outward. Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$. Closed

Div. Thm.

$$\iiint_E (2+5+3)\,dV = \iiint_E (10)\,dV$$

$= 10 \cdot$ (Volume of sphere with radius 3)

$$= 10 \cdot \frac{4}{3}\pi (3)^3 = 10 \cdot \frac{4}{3}\pi \cdot 27$$

$$= 10 \cdot 4\pi \cdot 9 = 360\pi$$

Great!

6. Show that for any vector field $\mathbf{F}(x, y, z)$ whose component functions have continuous partial derivatives, $\text{div}(\text{curl } \mathbf{F}) = 0$. Make it clear how the requirement that the partials be continuous is important.

$\vec{F} = \langle P, Q, R \rangle$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$\text{div}(\text{curl } \vec{F}) = \frac{\partial}{\partial x}(R_y - Q_z) + \frac{\partial}{\partial y}(P_z - R_x) + \frac{\partial}{\partial z}(Q_x - P_y)$$

$$= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz}$$

Rearrange to get:

$$P_{zy} - P_{yz} + Q_{xz} - Q_{zx} + R_{yx} - R_{xy}$$

By Clairaut's Theorem, we know that mixed partials are equivalent, given that the second order partial derivatives are continuous. Thus, we have that

$$P_{zy} = P_{yz}$$
$$Q_{xz} = Q_{zx}$$
$$R_{yx} = R_{xy}$$

Therefore, $\text{div}(\text{curl } \vec{F}) = 0$. $\square$

Nice!

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Crap. This line integral stuff is crazy. There's all these different ways, and at first I thought I'd just do them all the first way they showed us, so I kinda ignored a couple of days, but then it get pretty hard. They were saying something yesterday about a super-shortcut, where like, if the vector field was the right kind then for any closed path the answer is automatically zero. I like that, but what is it about the vector field that tells you when that happens?"

Help Biff by explaining as clearly as you can when you can tell that a line integral on any closed path has to come out to zero, and how you know.

Line integrals are sneaky in that way!
That special vector field they are talking about
is a conservative vector field, and no that doesn't
have anything to do with how it votes, just that it
has a potential function. The FTLI tells us
that when we have a conservative vector field
we can just compute the integral with the potential function
and the end points. But when the path is closed our
end point is the same as our beginning so the integral
will output 0

-Excellent!

8. Let $\mathbf{F}$ be the vector field $\mathbf{F}(x, y, z) = \langle x, y, 1\rangle$. Let S be the cylinder $x^2 + y^2 = 4$ between $z = 0$ and $z = 5$. Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

Long way

$\vec{r}(u,v) = \langle 2\cos u, 2\sin u, v\rangle$

$0 \le u \le 2\pi$

$0 \le v \le 5$

$\vec{F}(\vec{r}(u,v)) = \langle 2\cos u, 2\sin u, 1\rangle$

$\vec{r}_u = \langle -2\sin u, 2\cos u, 0\rangle$

$\vec{r}_v = \langle 0, 0, 1\rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin u & 2\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = (2\cos u)\vec{i} - (-2\sin u)\vec{j} + 0\vec{k}$$

$$= \langle 2\cos u, 2\sin u, 0\rangle$$

$$\int\int \langle 2\cos u, 2\sin u, 1\rangle \cdot \langle 2\cos u, 2\sin u, 0\rangle \, dS$$

$$= \int\int (4\cos^2 u + 4\sin^2 u) \, dS = \int_0^{2\pi}\int_0^5 4 \, dv\, du$$

$$= 40\pi.$$

Excellent!

9. Let $\mathbf{F}$ be the vector field $\mathbf{F}(x,y,z) = \langle xz, y^2, -x^3\rangle$. Let S be the portion of the paraboloid $z = x^2 + y^2$ below $z = 4$ with upward orientation. Find $\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S}$.

Stoke's Theorem!

$r(t) = \langle 2\cos t, 2\sin t, 4\rangle \quad 0 \le t \le 2\pi$

$\vec{F}(\vec{r}(t)) = \langle 8\cos t, 4\sin^2 t, -8\cos^3 t\rangle$

$r'(t) = \langle -2\sin t, 2\cos t, 0\rangle$

$$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \int_0^{2\pi} \langle 8\cos t, 4\sin^2 t, -8\cos^3 t\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle \, dt$$

$$= \int_0^{2\pi} -16\sin t\cos t + 8\cos t\sin^2 t \, dt$$

$u = \cos t$
$du = -\sin t\, dt$

$$= \int_0^{2\pi} -16\sin t\cos t\, dt + \int_0^{2\pi} 8\cos t\sin^2 t \, dt$$

$u = \sin t$
$du = \cos t$

$$= \int_0^{2\pi} 16u\, du \qquad = \int_0^{2\pi} 8u^2\, du$$

$$= 8\cos^2(t)\Big|_0^{2\pi} \qquad = \frac{8}{3}\sin^3(t)\Big|_0^{2\pi}$$

$$= 8\cos^2(2\pi) - 8\cos^2(0) \qquad = \frac{8}{3}\sin^3(2\pi) - \frac{8}{3}\sin^3(0)$$

$$= 8 - 8 \qquad = 0 - 0$$

$$= 0 \quad + \quad = 0$$

$$\boxed{= 0}$$

Wonderful!

10. Let $\mathbf{F}(x, y) = \langle 1, 0 \rangle$, and let C be the top half of a circle with some radius r centered at the origin, traversed counterclockwise from $(r, 0)$ to $(-r, 0)$. For what radius r will $\int_C \mathbf{F} \cdot d\mathbf{r} = 5$?

$r(t) = \langle r\cos t, r\sin t \rangle \quad 0 \le t \le \pi$

$F(\vec{r}(t)) = \langle 1, 0 \rangle$

$\vec{r}\,'(t) = \langle -r\sin t, r\cos t \rangle$

$$\int_C F \cdot d\vec{r} = \int_0^{\pi} \langle 1, 0 \rangle \cdot \langle -r\sin t, r\cos t \rangle \, dt$$

$$= \int_0^{\pi} -r\sin t \, dt$$

$$= r\cos(t) \Big|_0^{\pi}$$

$$= r\cos(\pi) - r\cos(0)$$

$$= r(-1) - r(1)$$

$$= -r - r$$

$$= -2r = 5$$

$$r = \frac{-5}{2}$$

negative radius not possible, thus we cannot have that $\int_C F \cdot d\vec{r} = 5$

Right