Exam 3 **Calc 1** **10/27/23**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. Find the derivative of $f(x) = \log_5 x + 3^x$.

$$\frac{1}{(\ln 5)x} + (\ln 3)3^x$$

Excellent!

$$(\log_a x)' = \frac{1}{(\ln a)x}$$

$$(a^x) = (\ln a)a^x$$

2. Find the derivative of $g(x) = \ln(x^3 + 1)$. chain rule

$$g'(x) = \frac{1}{x^3+1} \cdot 3x^2 = \frac{3x^2}{x^3+1}$$

- Great

3. Use the following table of values for $f(x)$ and $g(x)$ to find values for the following:

$\frac{1}{1+x^2}$

x	1	2	3	4	5	6
$f(x)$	5	4	6	1	3	2
$g(x)$	1	6	2	3	5	4
$f'(x)$	2	3	4	5	6	1
$g'(x)$	2	7	3	13	11	8

(a) If $h(x) = \tan^{-1}(f(x))$, what is $h'(3)$ and why?

$h'(x) = \frac{1}{1+f(x)^2} \cdot f'(x)$ $\quad (\tan^{-1}(x))' = \frac{1}{1+x^2}$ & Chain Rule

$h'(3) = \frac{1}{1+(6)^2} \cdot 4$

Good

(b) If $h(x) = g(x) \cdot \ln x$, what is $h'(5)$ and why? $\quad (\ln x)' = \frac{1}{x}$ & product Rule

$h'(x) = g'(x)\ln x + \frac{g(x)}{x}$

$h'(5) = (11)(\ln 5) + \frac{5}{5}$ or 1

(c) If $h(x) = \arcsin \frac{g(x)}{2}$, what is $h'(4)$ and why? $\sin^{-1}(x)$

$h'(x) = \frac{1}{\sqrt{1-\left(\frac{g(x)}{2}\right)^2}} \cdot \frac{g'(x)2 - g(x)\cdot 0}{2^2}$ $\quad (\sin^{-1})' = \frac{1}{\sqrt{1-x^2}}$

$h'(x) = \frac{1}{\sqrt{1-\left(\frac{g(x)}{2}\right)^2}} \cdot \frac{g'(x)}{2}$ $\quad$ & Chain & quotient Rule

$h'(4) = \frac{1}{\sqrt{1-\left(\frac{3}{2}\right)^2}} \cdot \frac{13}{2}$

4. Why is the derivative of $\ln x$ equal to $\frac{1}{x}$?

We know,

$$e^{\ln x} = x$$

Differentiating,

$$e^{(\ln x)} \cdot (\ln x)' = 1$$

or, $$(\ln x)' = \frac{1}{e^{(\ln x)}}$$

Excellent!

But, we also know that, $e^{\ln x} = x$

Therefore,

$$(\ln x)' = \frac{1}{x}$$

proved //

$\to \frac{\infty}{\infty}$ form Yes!

5. Evaluate $\lim_{x\to\infty} \frac{x^2}{e^x}$

$\underset{x\to\infty}{\text{L'Ham}} \frac{2x}{e^x}$ still $\frac{\infty}{\infty}$

$\underset{x\to\infty}{\text{L'Ham}} \frac{2}{e^x}$ closer & closer to 0

$\lim_{x\to\infty} \frac{2}{e^x} = 0$

Great!

6. Why is the derivative of $\arctan x$ equal to $\frac{1}{1+x^2}$?

$$\tan(\arctan x) = x$$

differentiate:

chain rule

$$\sec^2(\arctan x) \cdot (\arctan x)' = 1$$

$$(\arctan x)' = \frac{1}{\sec^2(\arctan x)}$$

$$x^2 + 1^2 = c^2$$

$$\sqrt{x^2+1} = c$$

$$\cos(\arctan x) = \frac{1}{\sqrt{x^2+1}}$$

$$\sec(\arctan x) = \frac{\sqrt{x^2+1}}{1}$$

$$\sec^2(\arctan x) = \left(\sqrt{x^2+1}\right)^2 = x^2+1$$

$$(\arctan x)' = \frac{1}{x^2+1}$$

Great

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "OMG! This is literally impossible for anybody to ever pass! You learn something, but then the next day they tell you it's not right anymore! So, I got it when, like, the derivative of ln x was 1 over x, right? But then they had us do the derivative of ln 10, so I said 1 over 10, right? But they said it wasn't right anymore!"

Help Bunny understand what's going on.

So, the derivative of $\ln x$ is $\frac{1}{x}$, which is totally correct but for the case of $\ln(10)$, it's a constant. And as all the other constants, the derivative of $\ln(10)$ is also zero. Therefore, the derivative of $\ln(10)$ cannot be $\frac{1}{10}$.

Good

0, 54 2, 40

8. A can of Mtn Dew contains 54 mg of caffeine. Suppose that amount enters the bloodstream immediately. If 40mg remains in the bloodstream 2 hours after drinking,

(a) Find a function of the form $f(x) = Ab^x$ for the amount in the bloodstream after x hours have passed.

$54 = A \cdot b^0$

$\underline{54 = A}$

$40 = 54 \cdot b^2$

$\frac{40}{54} = b^2$

$\frac{40}{54} = b^2$ $\underline{b = 0.860606}$

$\underline{f(x) = 54 \cdot .860606^x}$

(b) What does your formula project the amount in the bloodstream will be after 5 hours?

$f(5) = 54 \cdot 0.860606^5$

$\underline{= 25.50068554}$

(c) At what rate does your formula predict the amount of caffeine in the bloodstream will be changing after 5 hours?

$f'(x) = 54(\ln 0.860606)(0.860606)^x$

$f'(5) = 54(\ln 0.860606)(0.860606)^5$

$\underline{= -3.8265}$ *negative because # is going down in bloodstream!

9. Evaluate $\lim_{x \to 0} \dfrac{\sin^{-1} x}{\sqrt{x}}$.

$\frac{0}{0}$ indeterminate form Yes!

$$= \lim_{x \to 0} \frac{\sin^{-1} x}{x^{1/2}}$$

$$\overset{L'H}{=} \lim_{x \to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{\frac{1}{2} x^{-1/2}}$$

$$= \lim_{x \to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{\frac{1}{2\sqrt{x}}}$$

multiply by reciprocal

$$\lim_{x \to 0} \frac{2\sqrt{x}}{\sqrt{1-x^2}}$$

$$= \frac{2(\sqrt{0})}{\sqrt{1-0^2}}$$

Excellent!

$$= \frac{0}{1} = 0$$

10. Let $f(x) = \frac{1}{2-x}$.

(a) Find $f^{-1}(x)$.

$$y = \frac{1}{2-x}$$

$$(2-y)\, x = \frac{1}{\cancel{2-y}} \cdot \cancel{2-y}$$

$$2 - y = \frac{1}{x} - 2$$

$$-2$$

$$\boxed{y = \frac{-1}{x} + 2}$$

(b) Find $f'(5)$ and $(f^{-1})'(-\frac{1}{3})$

$$f'(5) = (2-x)^{-1}$$

$$f'(5) = -1(2-x)^{-2} \cdot -1$$

$$f'(5) = \frac{1}{(2-5)^2}$$

$$\boxed{f'(5) = \frac{1}{9}}$$

$$(f^{-1})'\left(\tfrac{1}{3}\right) = \frac{1}{x^2}$$

$$(f^{-1})\left(\tfrac{1}{3}\right) = \frac{1}{\frac{1}{9}} = 9$$

$$\boxed{(f^{-1})'\left(\tfrac{1}{3}\right) = 9}$$

(c) What's the connection between the answers to part b, and why?

They are the reciprocals of each other. and that makes sense. since they're inverses.

yes!