**Exam 2** **Calc 3** **10/27/23**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. Write a double Riemann sum for $\iint_R f\, dA$, where $R = \{(x, y) : 2 \le x \le 6, 0 \le y \le 4\}$ using midpoints with $n = m = 2$ subdivisions


$\Delta x = 2$

$\Delta y = 2$

$\Delta x \cdot \Delta y = 4$

$4 \times (f(3,1) + f(3,3) + f(5,1) + f(5,3))$

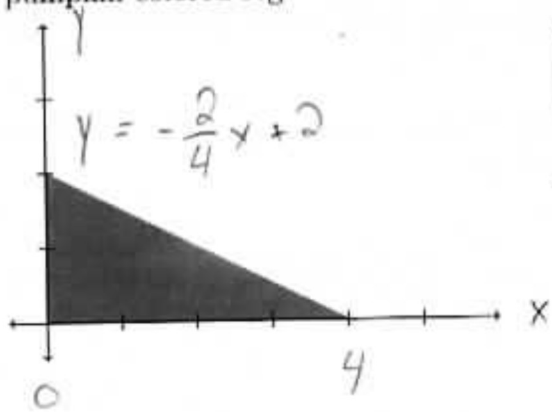
Good

2. Set up a double integral for the integral from #1.

$\int_0^4 \int_2^6 f \; dx\, dy$.

Yep

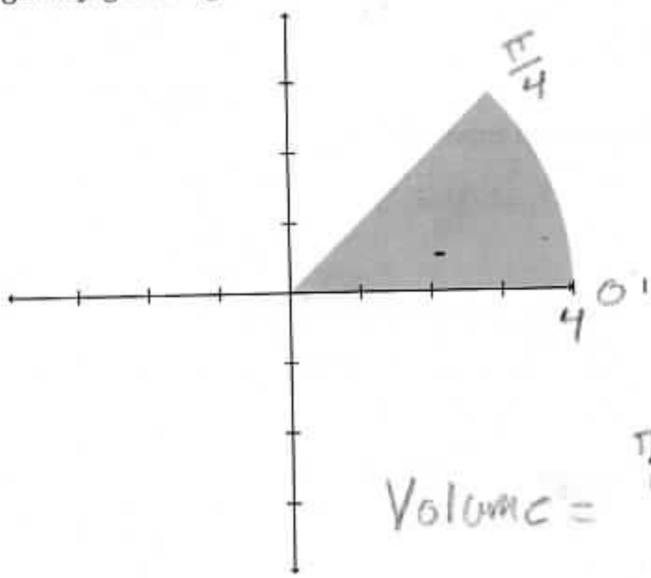
3. Set up limits of integration for finding the volume under $f(x, y) = 3 + y$ within the pumpkin-colored region shown:




$$\int_0^4 \int_0^{-\frac{2}{4}x+2} 3+y \, dy\,dx$$

Good

4. Set up limits of integration for finding the volume under $g(x, y) = 10 - x$ within the ghostly green region shown:

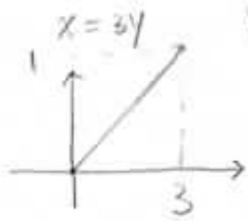



Great

$$\text{Volume} = \int_0^{\pi/4} \int_0^4 \int_0^{10-x} r\,dz\,dr\,d\theta$$

$$= \int_0^{\pi/4} \int_0^4 \int_0^{10-r\cos\theta} r\,dz\,dr\,d\theta = \int_0^{\pi/4} \int_0^4 (10-r\cos\theta)\,r\,dr\,d\theta$$

5. Evaluate $\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy$




$y = \frac{1}{3}x$

$y = 1$

$$\int_0^3 \int_0^{\frac{1}{3}x} e^{x^2}\,dy\,dx$$

$$= \int_0^3 e^{x^2} \cdot y \Big|_0^{\frac{1}{3}x}\,dx$$

$$= \int_0^3 \frac{1}{3}x \cdot e^{x^2}\,dx$$

let $u = x^2$

$$\frac{du}{dx} = 2x$$

$$dx = \frac{du}{2x}$$

$$= \int_0^3 \frac{1}{3}\cancel{x} \cdot e^u \frac{du}{2\cancel{x}}$$

$$= \frac{1}{6}\int_0^3 e^u\,du$$

$$= \frac{1}{6}\, e^{x^2}\Big|_0^3$$

$$= \frac{1}{6}\left(e^9 - e^0\right)$$

$$= \frac{1}{6}\left(e^9 - 1\right)$$

Excellent!

6. Show that the Jacobian for the conversion from rectangular to polar coordinates is what it is.

$x = r\cos\theta$

$y = r\sin\theta$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = \cos\theta \cdot r\cos\theta - \sin\theta(-r\sin\theta)$$

$$= r\cos^2\theta + r\sin^2\theta$$

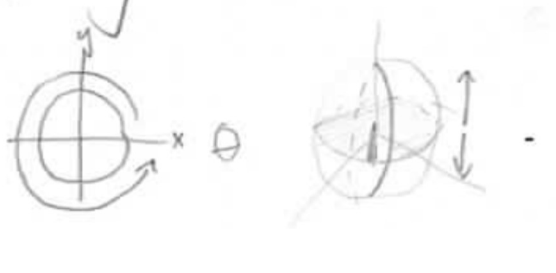
$$= r(\underbrace{\cos^2\theta + \sin^2\theta}_{1}) = r$$

Excellent!

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "OMG! These spherical thingies are so hard! I tried to do, like, the online homework, right? And I put limits of 0 to $2\pi$ on everything with angles because this one guy who was working in the computer lab told me that's pretty much always right? And the theta ones it was right on a lot of them, but on the phi ones it wasn't right on *any* of them! Can you believe how unfair that is?"

Help Bunny by explaining as clearly as you can why the responses she got make sense.

Well Bunny, when you integrate in spherical coordinates you use theta = $2\pi$ to go all the way around the circle (usually in the xy plane). Therefore, you've essentially covered the x and y axes in terms of degrees. Since the only axis left is the z-axis, you only need to go from the top ($\phi = 0$) to the bottom ($\phi = \pi$) in order to get the whole range of the solid in the z-axis. If you go all the way around, $0$-$2\pi$, you are essentially going top-to-bottom-to-top again, and doubling the volume.




Great

8. Set up integrals for the $x$ coordinate of the center of mass of the first-octant portion of a sphere with radius 5 and uniform density.

$X = \rho \sin\phi \cos\theta$

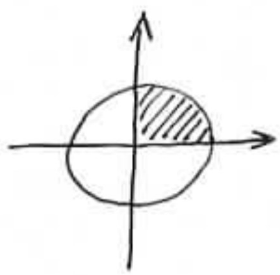
$$\bar{X} = \frac{\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{5} (\rho\sin\phi\cos\theta)(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta}{\displaystyle\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{5} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$$

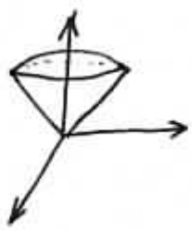
Good

9. Evaluate the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z \, dz \, dy \, dx$.

$$z = \sqrt{2 - x^2 - y^2} \Rightarrow z^2 = 2 - x^2 - y^2 \Rightarrow \overbrace{x^2 + y^2 + z^2 = (\sqrt{2})^2}^{\text{sphere w/ } r = \sqrt{2}}$$

$$z = \sqrt{x^2 + y^2} \Rightarrow z^2 = x^2 + y^2 \quad \text{cone with } \phi = \frac{\pi}{4}$$





$$\iiint_E z \, dV = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{2}} (\rho \cos\phi) \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{2}} \rho^3 \sin\phi \cos\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \frac{\rho^4}{4} \sin\phi \cos\phi \Big|_0^{\sqrt{2}} d\phi \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \sin\phi \cos\phi \, d\phi \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2} (\sin\phi)^2 \Big|_0^{\frac{\pi}{4}} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{4} \, d\theta$$

$$= \frac{1}{4} \theta \Big|_0^{\pi/2}$$

$$= \frac{\pi}{8}$$

10. Set up an integral for the volume of the solid between $z = x^2 + y^2$ and $z = x + 2$.

Intersection? $x^2 + y^2 = x + 2$

$x^2 - x + y^2 = 2$ } Looks like an ellipse!

$y^2 = 2 - x^2 + x$

$y = \pm\sqrt{2 - x^2 + x}$

When $y = 0$:

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2$ or $x = -1$

$$\text{Volume} = \int_{-1}^{2} \int_{-\sqrt{2-x^2+x}}^{\sqrt{2-x^2+x}} \int_{x^2+y^2}^{x+2} 1 \, dz\, dy\, dx$$