Exam 2b **Calc 1** **10/3/25**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. State the formal definition of the derivative of a function $f(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Good

2. [WeBWorK] Find an equation for the line tangent to the graph of

$$f(x) = -5xe^x \qquad -15 \cdot e^3$$

at the point $(a, f(a))$ for $a = 3$.

Product Rule

$(3, f(3)) \rightarrow (3, -15e^3)$ point for pt

$-5 \cdot e^x + -5x \cdot e^x$

slope line (Derivative)

$f'(x) = -5e^x - 5xe^x$

$-5e^3 - 15e^3$ $\quad -20e^3$ Plug in $x=3$ for slope

$y - (-15e^3) = -20e^3(x-3)$ Build equation and simplify

Excellent!

$y + 15e^3 = -20e^3 x + 60e^3$

$y = -20e^3 x + 45e^3$

3. [WeBWorK] Let $f(x) = \frac{5}{3x+6}$. Find $f'(x)$

$5(3x+6)^{-1}$ $\quad 5x^{-1}$

$-5(3x+6)^{-2} \cdot 3$ $\quad -5x^{-2}$

$f'(x) = -15(3x+6)^{-2}$

$f'(x) = \frac{-15}{(3x+6)^2}$

Good

4. Use the definition of the derivative to find the derivative of $f(x) = \sqrt{x}$.

the derivative function

is $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$

$f'(x) \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})}$

$f'(x) = \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$

$f'(x) \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$

$f'(x) = \frac{1}{\sqrt{x+0} + \sqrt{x}}$

Excellent!

$f'(x) = \frac{1}{\sqrt{x} + \sqrt{x}}$

$f'(x) = \frac{1}{2\sqrt{x}}$

5. Show why the derivative of $\cot x$ is $-\csc^2 x$.

$\cot = \frac{\cos(x)}{\sin(x)}$

$\sin^2(x) + \cos^2(x) = 1$

$f'(x) = \frac{(-\sin(x))(\sin(x)) - (\cos(x))(\cos(x))}{\sin^2(x)}$

$= \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)}$

$= \frac{-(\sin^2(x) + \cos^2(x))}{\sin^2(x)}$

Excellent!

$= \frac{-1}{\sin^2(x)}$

$= -\csc^2(x)$

6. Prove the Product Rule for derivatives.

$$[f(x)\cdot g(x)]' = \lim_{h\to 0} \frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h}$$

add zero

$$= \lim_{h\to 0} \frac{f(x+h)\cdot g(x+h) - g(x+h)\cdot f(x) + f(x)\cdot g(x+h) - f(x)\cdot g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\cdot g(x+h) + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}\cdot f(x)$$

because $g(x)$ and $f(x)$ are differentiable

$$= \lim_{h\to 0} f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

Excellent!

7. Sam is a calculus student at Enormous State University, and they're having some trouble with derivatives. Sam says "Wow, this derivative stuff is confusing. What really gets me is the product and quotient and chain stuff. So like, we're supposed to work in groups in class, which is sad because none of us know how to do it so we're just confused together. But one person said for sine squared of x we should do the chain rule, and this other person said we should write it as sine times sine and use the product rule. How do you tell which way is right?"

   Help Sam by explaining, as clearly as possible, how to decide between their options.

$(\sin x)^2$ $\quad 2(\sin x)\cdot\cos x$

$(\sin x)(\sin x)$

$(\cos x)(\sin x)+(\sin x)(\cos x)$

OH HEY THERE SAM! THE CHAIN RULE, PRODUCT RULE, + QUOTIENT RULE ARE REALLY CONFUSING! THE WAY TO DECIDE WHICH ONE IS RIGHT TO USE FOR A QUESTION LIKE $(\sin x)^2$ IS WHICHEVER ONE YOU FEEL MOST COMFORTABLE USING! THIS IS BECAUSE $(\sin x)^2$ AND $(\sin x)(\sin x)$ ARE THE SAME THING, WHETHER YOU USE THE CHAIN OR PRODUCT RULE, YOU GET THE SAME THING! HERE!

Great

PRODUCT RULE

$(\sin x)(\sin x) = (\cos x)(\sin x)+(\sin x)(\cos x)$

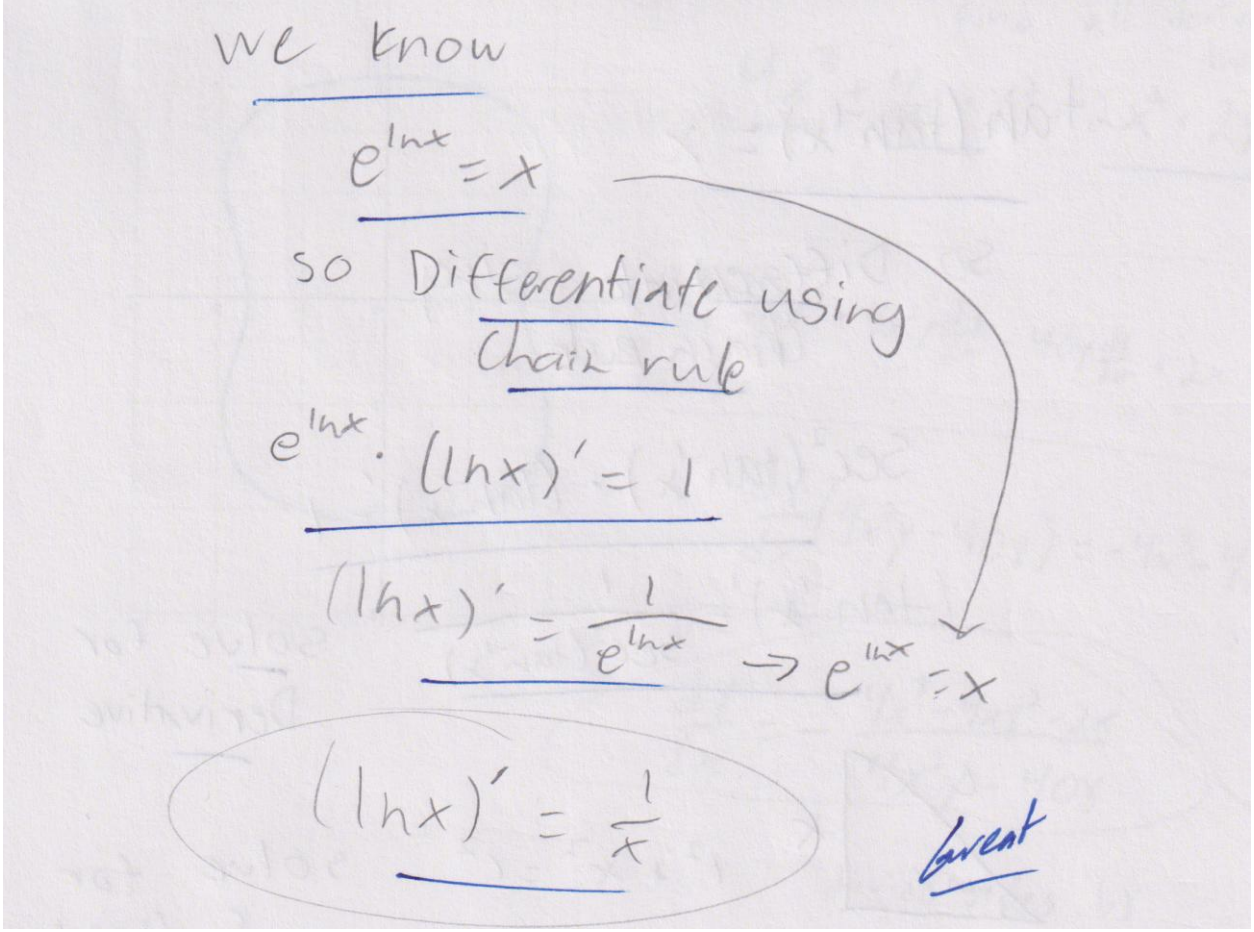
$= 2\sin x\cos x$

CHAIN RULE

$(\sin x)^2 \rightarrow = 2(\sin x)\cdot(\cos x)$

$= 2\sin x\cos x$

8. Show why the derivative of $\ln x$ is $1/x$.



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9. Show why the derivative of $\tan^{-1} x$ is $\dfrac{1}{1+x^2}$.

$\tan(\tan^{-1} x) = x$

$\sec^2(\tan^{-1} x) \cdot (\tan^{-1} x)' = 1$

$(\tan^{-1} x)' = \dfrac{1}{\sec^2(\tan^{-1} x)}$

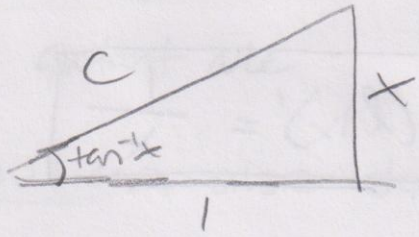
$= \dfrac{1}{\frac{1}{\cos^2(\tan^{-1} x)}}$

$= \cos^2(\tan^{-1} x)$

$= [\cos(\tan^{-1} x)]^2$

$= \left(\dfrac{1}{\sqrt{1+x^2}}\right)^2$

$= \boxed{\dfrac{1}{1+x^2}}$



$1^2 + x^2 = c^2$

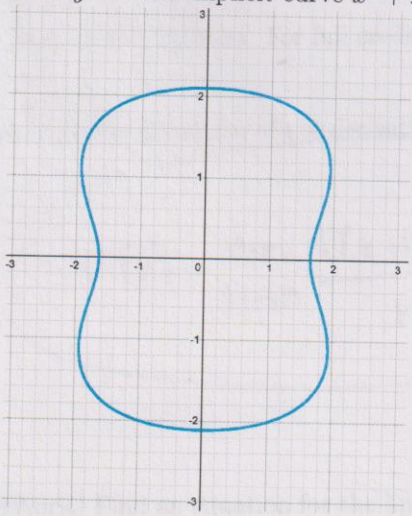
$\sqrt{c^2} = \sqrt{1+x^2}$

$c = \sqrt{1+x^2}$

Excellent!

$4x \cdot y2 + 2x2 \cdot 2y \cdot y' + 20y3 \cdot y'$

10. (a) Find y' for the implicit curve $x^4 + 2x^2y^2 + 5y^4 + x^2 - 20y^2 = 13$.



$$y' = \frac{-(4x^3 + 4xy^2 + 2x)}{2x^2 2y + 20y^3 - 40y}$$

$y' = \frac{1}{4}$ AFTER PLUGGING IN $(1, -2)$

Nice Work!

$4x^3 + 4x \cdot y2 + 2x2 \cdot 2y \cdot y' + 20y^3 \cdot y' + 2x - 40y \cdot y' = 0$

$4x^3 + 4xy2 + 2x2 2y \cdot y' + 20y^3 y' + 2x - 40y \cdot y' = 0$

$-4x^3 \quad -4xy2 \qquad\qquad\qquad\qquad -2x$

$2x^2 2yy' + 20y^3 y' - 40yy' = -(4x^3 + 4xy2 + 2x)$

$y'(2x^2 2y + 20y^3 - 40y) = -(4x^3 + 4xy2 + 2x)$

(b) What is the slope of the tangent line to the curve from part (a) at the point $(1, -2)$?

$$y' = \frac{-(4(1)^3 + 4(1)(-2)^3 + 2(1))}{2(1)^2 2(-2) + 20(-2)^3 - 40(-2)}$$

$y' = 1/4$