Exam 4 Sample **Calc 1** **11/14/25**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. State the definition of the definite integral of a function $f(x)$ on the interval $[a, b]$.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot\Delta x$$

2. Evaluate $\displaystyle\int_1^4 \sqrt{x}\,dx$ exactly.

$$\int_1^4 x^{1/2}\,dx = \frac{2}{3}\cdot x^{3/2}\Big|_1^4$$

$$= \frac{2}{3}\cdot 4^{3/2} - \frac{2}{3}\cdot 1^{3/2}$$

$$= \frac{2}{3}\cdot 8 - \frac{2}{3}\cdot 1$$

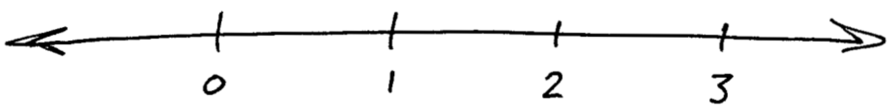
$$= \frac{16}{3} - \frac{2}{3}$$

$$= \frac{14}{3}$$

3. The following table of values for $v(t)$ gives the position (in feet beyond the point where the car was at time 0) of a car t seconds after the driver hits the brakes because they saw a glowing purple giraffe.

t	$v(t)$
0.0	88
0.5	78
1.0	56
1.5	37
2.0	19
2.5	7
3.0	0

Find L_3, R_3, and M_3, the left-hand, right-hand, and midpoint approximations for the distance travelled by the car over the three seconds it takes to come to a stop.



$L_3 = f(0)\cdot 1 + f(1)\cdot 1 + f(2)\cdot 1$

$= 88\cdot 1 + 56\cdot 1 + 19\cdot 1$

$= 163$ ft

$R_3 = f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1$

$= 56\cdot 1 + 19\cdot 1 + 0\cdot 1$

$= 75$ ft

$M_3 = f(0.5)\cdot 1 + f(1.5)\cdot 1 + f(2.5)\cdot 1$

$= 78\cdot 1 + 37\cdot 1 + 7\cdot 1$

$= 122$ ft

4. If $\int_0^7 f(x)\,dx = 12$, $\int_0^5 f(x)\,dx = 3$, and $\int_3^5 f(x)\,dx = 4$, what is $\int_5^7 f(x)\,dx$ and why?

$$\int_0^5 f(x)dx + \int_5^7 f(x)dx = \int_0^7 f(x)dx, \text{ so}$$

$$3 + \int_5^7 f(x)dx = 12$$

$$\int_5^7 f(x)dx = 12 - 3 = 9$$

It's because the integral from 0 to 7 combines the part from 0 to 5 with the part from 5 to 7, so we can work backwards to see how much was from the 5 to 7 range.

5. A ball is thrown into the air by a baby alien on a planet in the system of Alpha Centauri with an initial velocity of 30 ft/s. The acceleration due to gravity there is -54 ft/s^2. Find the ball's height after 1 second.

$v(t) = -54t + C$

antiderivative $a(t) = -54$

Then since $v(0) = 30$,

$v(0) = -54(0) + C = 30$

so $C = 30$

So from $v(t) = -54t + 30$ we can antidifferentiate to get

$h(t) = -27t^2 + 30t + D$ where the D might as well be 0 for the height it was thrown from.

Then we have $h(t) = -27t^2 + 30t$

So $h(1) = -27(1)^2 + 30(1)$

$= -27 + 30$

$= 3$

So after 1 second the ball is 3 ft above the height it was thrown from.

6. Suppose that $f''(x) = 3\sin x$, $f'(0) = 0$, and $f(0) = 3$. Find $f(x)$.

Since $f''(x) = 3\sin x$ we can antidifferentiate to get

$$f'(x) = -3\cos x + C$$

Because we know $f'(0) = 0$,

$$0 = f'(0) = -3\cos 0 + C$$

$$0 = -3 + C$$

$$C = 3$$

So our new improved $f'(x)$ is

$$f'(x) = -3\cos x + 3$$

and antidifferentiating:

$$f(x) = -3\sin x + 3x + D$$

Now since $f(0) = 3$,

$$3 = f(0) = -3\sin(0) + 3(0) + D$$

$$D = 3$$

Then our ~~final version is~~

$$f(x) = -3\sin x + 3x + 3$$

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Crap. This Calculus stuff is pretty rough. So, like, sometimes I get negative numbers when I do the definite integrate things, right? So the answer in the back of the book is pretty much always just what I got but with the negative taken off, right? So I heard it's like that because you sometimes get things upside down, like with the bottom thing first or whatever, right? So do you always just take the negative sign off?"

Help Biff by explaining as clearly as you can whether his reasoning holds, or if there are limitations.

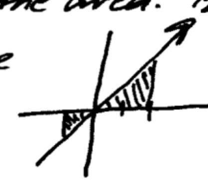
So Biff, <u>sometimes</u> just taking the negative sign off is great, but not always. You should understand about why.

If you think about $\int_0^2 x\,dx$, the picture looks like

The answer comes out to 2, the area of that triangle.

But for $\int_{-1}^{0} x\,dx$ the picture looks like

and the answer comes out to $-\frac{1}{2}$. The actual area is $\frac{1}{2}$, but the negative is because it's below the x-axis. So that time just taking the negative off gets the area. But if we do $\int_{-1}^{2} x\,dx$ then the picture looks like



The integral comes out to $\frac{3}{2}$, and neither that nor the negative of that are the area.

It could represent something like losing money for a while and then making money, or draining and then charging your phone. You can't just use your phone for hours and then take the negative sign off and say you charged it all that time. Lots of times those opposite directions <u>matter</u>.

8. Let $f(x) = \int_{\pi/2}^{x} \cos(t^2)\, dt$. What is $f'(x)$?

By Part I of the Fun. Theorem of Calculus,

$$\frac{d}{dx}\int_{\pi/2}^{x} \cos(t^2)\,dt = \cos(x^2)$$

9. Evaluate $\int \sqrt{4x+3}\, dx$.

Let $u = 4x+3$

$$\frac{du}{dx} = 4$$

$$du = 4dx$$

$$\frac{du}{4} = dx$$

$$= \int u^{1/2} \cdot \frac{du}{4}$$

$$= \frac{1}{4} \cdot \frac{2}{3} \cdot u^{3/2} + C$$

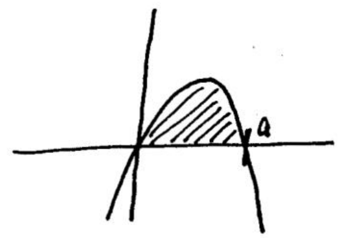
$$= \frac{1}{6}(4x+3)^{3/2} + C$$

10. Let a be some positive constant. Find the area of the region under the graph of $y = x(a - x)$ and above the x-axis.

I need to know where $y = x(a-x)$ intersects the x-axis, which is $y=0$, so I set those equal to each other, or

$$0 = x(a-x) \Rightarrow x = 0 \text{ or } x = a$$

Trying the graph on my calculator for a few different a values shows me it looks like



So $\int_0^a x(a-x)\,dx = \int_0^a (ax - x^2)\,dx$

$$= \left[\frac{ax^2}{2} - \frac{x^3}{3}\right]_0^a$$

$$= \left[\frac{a(a)^2}{2} - \frac{(a)^3}{3}\right] - \left[\frac{a(0)^2}{2} - \frac{(0)^3}{3}\right]$$

$$= \frac{a^3}{2} - \frac{a^3}{3}$$

$$= \frac{a^3}{6}$$