Exam 1 **Calc 3** **9/27/25**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. State the definition of the partial derivative of a function $f(x, y)$ with respect to x.

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

Good!

2. Show that the function $f(x, y) = \dfrac{xy + y^2}{x^2 + y^2}$ fails to have a limit at $(0, 0)$.

approaching from $x = \emptyset$

$$\lim_{y \to \emptyset} \frac{\emptyset y + y^2}{\emptyset^2 + y^2} = \lim_{y \to \emptyset} \frac{y^2}{y^2} = 1$$

approaching from $y = \emptyset$

$$\lim_{x \to \emptyset} \frac{x(\emptyset) + \emptyset^2}{x^2 + \emptyset^2} = \lim_{x \to \emptyset} \frac{\emptyset}{x^2} = \emptyset$$

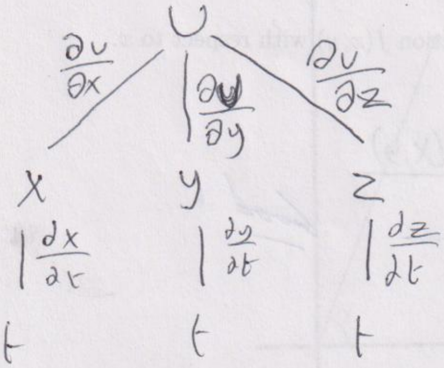
approaching from $y = x$

$$\lim_{x \to \emptyset} \frac{x(x) + x^2}{x^2 + x^2} = \lim_{x \to \emptyset} \frac{2x^2}{2x^2} = 1$$

Because the limits of $f(x,y)$ as it approaches $(0,0)$ from different directions are not equal, the overall limit of $f(x,y)$ as it approaches $(0,0)$ DOES NOT EXIST

Excellent!

3. Suppose that $u = f(x, y, z)$, where $x = x(t), y = y(t)$, and $z = z(t)$. Write the Chain Rule formula for $\frac{du}{dt}$. Make very clear which derivatives are partials.



$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

Partials: $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$

Great

4. Find the directional derivative of $f(x, y) = 7x - x^3y + y^3$ at the point $(-1, 2)$ in the direction of the vector $\langle -3, 4 \rangle$.

$\boxed{D_{\vec{u}} f(x_0, y_0) = a f_x(x_0, y_0) + b f_y(x_0, y_0)}$

$\vec{u} = \frac{\langle -3, 4 \rangle}{\sqrt{-3^2 + 4^2}}$ $\quad 9 \quad 16$

$\vec{u} = \langle \frac{-3}{5}, \frac{4}{5} \rangle$

$f_x(x, y) = 7 - 3x^2y \longrightarrow f_x(-1, 2) = 7 - 3(-1)^2(2)$

$= 1$

$f_y(x, y) = -x^3 + 3y^2$

$\longrightarrow f_y(-1, 2) = -(-1)^3 + 3(2)^2$

$1 + 12$

$= 13$

$D_{\vec{u}} f(-1, 2) = \frac{-3}{5}(1) + \frac{4}{5}(13)$

$= -\frac{3}{5} + \frac{52}{5}$

$= \boxed{\frac{49}{5}}$

Good

5. Let $f(x,y) = \dfrac{x}{3x+2y}$.

(a) In which direction is the directional derivative greatest at the point $(-1,3)$?

in direction of gradient, $\nabla f(x,y)$

$$\nabla f(x,y) = \left\langle \frac{3x+2y-3x}{(3x+2y)^2}, \frac{-2x}{(3x+2y)^2} \right\rangle = \left\langle \frac{2y}{(3x+2y)^2}, \frac{-2x}{(3x+2y)^2} \right\rangle$$

$$\nabla f(x,y)\Big|_{(-1,3)} = \left\langle \frac{2(3)}{(3(-1)+2(3))^2}, \frac{-2(-1)}{(3(-1)+2(3))^2} \right\rangle$$

Great

$$= \left\langle \frac{6}{9}, \frac{2}{9} \right\rangle$$

in the direction of $\left\langle \frac{2}{3}, \frac{2}{9} \right\rangle$

(b) What is the value of the directional derivative in the direction of the vector from part (a)?

$$D_r D_r = \|\nabla f(x,y)\| \cancel{\cos\theta}$$

Yes.

$$1$$

$$\left\| \left\langle \frac{2}{3}, \frac{2}{9} \right\rangle \right\|$$

$$= \sqrt{\frac{4}{9} + \frac{4}{81}}$$

$$= \sqrt{\frac{36}{81} + \frac{4}{81}} = \sqrt{\frac{40}{81}} = \boxed{\frac{2\sqrt{10}}{9}}$$

6. Show that for any vectors $\vec{a}$ and $\vec{b}$ the vector $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$.

let $\vec{a} = \langle a_1, a_2, a_3 \rangle$

let $\vec{b} = \langle b_1, b_2, b_3 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i(a_2 b_3 - a_3 b_2) - j(a_1 b_3 - a_3 b_1) + k(a_1 b_2 - a_2 b_1)$$

$$\langle a_2 b_3 - a_3 b_2, -a_1 b_3 + a_3 b_1, a_1 b_2 - a_2 b_1 \rangle \cdot \langle a_1, a_2, a_3 \rangle$$

$$= a_1 a_2 b_3 - a_1 a_3 b_2 - a_2 a_1 b_3 + a_2 a_3 b_1 + a_3 a_1 b_2 - a_3 a_2 b_1 = \emptyset$$

Because $(\vec{a} \times \vec{b}) \cdot \vec{a} = \emptyset$, this means $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$

Excellent!

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Dude! This Calc stuff is totally not even math! The professor asked us, like, what the gradient means. Like, is this philosophy or something? It means you do the f-ing partials, right? Its direction is what the f-ing partials say! It doesn't mean anything, it's just math!"

Explain clearly to Biff what the direction of the gradient means, and how you know that.

Firstly, the gradient $\nabla f(x,y) \equiv \langle f_x(x,y), f_y(x,y) \rangle$

It's the rate of change in each direction x and y combined into a vector to describe the total change of the "z" $f(x,y)$.

Where does this vector point?

The gradient points to the steepest slope for whatever point you plug into it that lies in $f(x,y)$.

Since the directional derivative is equal to $\|\nabla f(x,y)\| \cos\theta$, the greatest this value can be is in the direction of the gradient. Pointing "in the gradient" is saying you turn zero degrees, or $\cos\theta = 1$, so the gradient is maximized.

Thus the "direction of the gradient" is a way to say "the steepest slope for the point (a,b) on the function $f(x,y)$"

Lagrange

8. Find the maximum and minimum **values** of $f(x, y) = 4x+y$ on the ellipse $x^2+49y^2 = 1$.

$\nabla f = \langle 4, 1 \rangle$ $\qquad \nabla g = \langle 2x, 98y \rangle$

$\langle 4, 1 \rangle = \langle 2x, 98y \rangle \lambda$

$4 = 2x\lambda$

$\frac{4}{2\lambda} = x$

$\frac{2}{\lambda} = x$

$1 = 98y\lambda$

$\frac{1}{98\lambda} = y$

$\boxed{\pm \frac{1}{\sqrt{38465}} = y}$

$\left(\frac{2}{\lambda}\right)^2 + 49\left(\frac{1}{98\lambda}\right)^2 = 1$

$\frac{4}{\lambda^2} + \frac{49}{9604\lambda^2} = 1$

$\frac{1}{\lambda^2}\left(4 + \frac{49}{9604}\right) = 1$

$4 + \frac{49}{9604} = \lambda^2$

$\sqrt{\frac{38465}{9604}} = \sqrt{\lambda^2}$

$\pm \frac{\sqrt{38465}}{98} = \lambda$

$\frac{\frac{2}{\sqrt{38465}}}{98} = x$

$\boxed{\pm \frac{196}{\sqrt{38465}} = x}$

Nice!

$f\left(\frac{196}{\sqrt{38465}}, \frac{1}{\sqrt{38465}}\right) = 4\left(\frac{196}{\sqrt{38465}}\right) + \left(\frac{1}{\sqrt{38465}}\right) = \boxed{\frac{785}{\sqrt{38465}}}$ max

$f\left(-\frac{196}{\sqrt{38465}}, -\frac{1}{\sqrt{38465}}\right) = 4\left(\frac{-196}{\sqrt{38465}}\right) + \left(-\frac{1}{\sqrt{38465}}\right) = \boxed{\frac{-785}{\sqrt{38465}}}$ min

$196 \times 4 = 784$

9. Find and classify all extrema of the function $f(x,y) = \frac{x^2}{2} + 2y^3 + 3y^2 - xy - x$ and classify each as a maximum, minimum, or saddle point.

$D\,F(x,y) = f_{xx}(x,y) f_{yy}(x,y) - (f_{xy}(x,y))^2$

$f_x(x,y) = x - y - 1$ $\qquad$ $f_{xx} = 1$

$f_y(x,y) = 6y^2 + 6y - x$ $\qquad$ $f_{yy} = 12y + 6$

$f_{xy}(x,y) = -1$

$x - y - 1 = 0$ $\qquad$ $y = x - 1$

$6y^2 + 6y - x = 0$

$6(x-1)^2 + 6(x-1) - x = 0$

$6(x^2 - 2x + 1) + 6x - 6 - x = 0$

$6x^2 - 7x = 0$

$x(6x - 7) = 0$

$x = 0$ $\qquad$ $x = \frac{7}{6}$

if $x = 0$

$y = -1$ $\qquad$ if $x = \frac{7}{6}$

Excellent.

$y = \frac{1}{6}$

$(0,-1)$ and $(\frac{7}{6}, \frac{1}{6})$

$D\,f(0,-1) = (1)(12(-1)+6) - (-1)^2 = -7 < 0$ $\qquad$ $(0,-1) \to$ saddle

$D\,f(\frac{7}{6}, \frac{1}{6}) = (1)(12(\frac{1}{6})+6) - (-1)^2 = 7 > 0$ min or max

f_{xx} is > 0 so

$(\frac{7}{6}, \frac{1}{6}) \to$ min

10. Suppose you're standing at the point $(-1, 3)$ on the surface $f(x, y) = \dfrac{x}{3x + 2y}$. In some directions you face, you'll be looking uphill. In other directions, you'll be facing downhill. In which directions are you facing uphill?

Facing uphill at gradient and up to 90° in either direction (excluding final end points of 90° and −90°)

f_x and f_y taken from earlier problem

$$f_x = \frac{2}{3}$$

$$\text{gradient} = \left\langle \frac{2}{3}, \frac{2}{9} \right\rangle$$

$$f_y = \frac{2}{9}$$

Will be looking up at gradient direction of $\left\langle \frac{2}{3}, \frac{2}{9} \right\rangle$ and up to but not including 90° in either direction. Exactly 90° away will be the level ~~curv~~ ~~cuv~~ curve where the direction switches from uphill to downhill.

Excellent