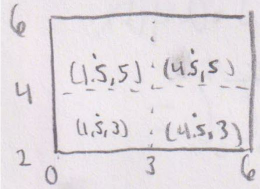
Exam 2 **Calc 3** **10/24/25**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. Write a double Riemann sum for $\iint_R f\,dA$, where $R = \{(x, y) : 0 \le x \le 6, 2 \le y \le 6\}$ using midpoints with $n = m = 2$ subdivisions



$\Delta x = 3 \quad \Delta y = 2 \quad \Delta A = 3 \cdot 2 = 6$

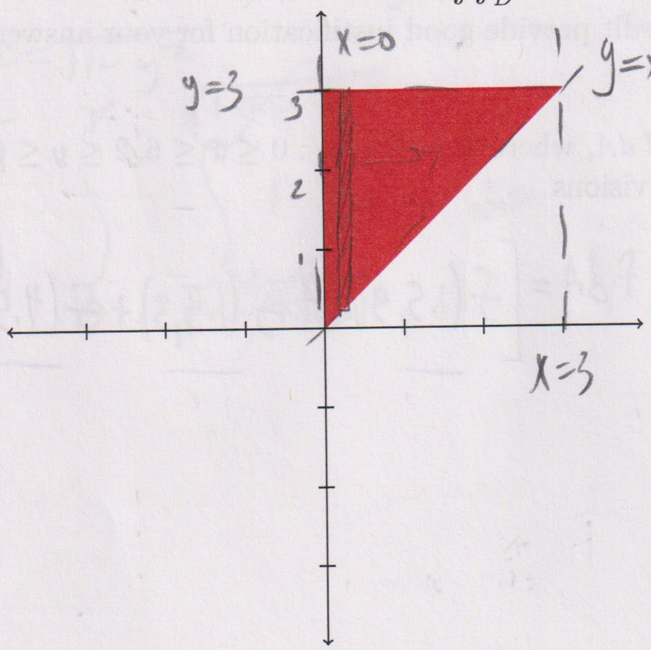
$6(f(1.5,3) + f(4.5,3) + f(1.5,5) + f(4.5,5))$

Great

2. Set up a double integral for the integral from #1.

$$\int_0^6 \int_2^6 f \, dy\, dx$$

3. Set up limits of integration for $\iint_D f(x,y)\,dA$ over the region shown below.

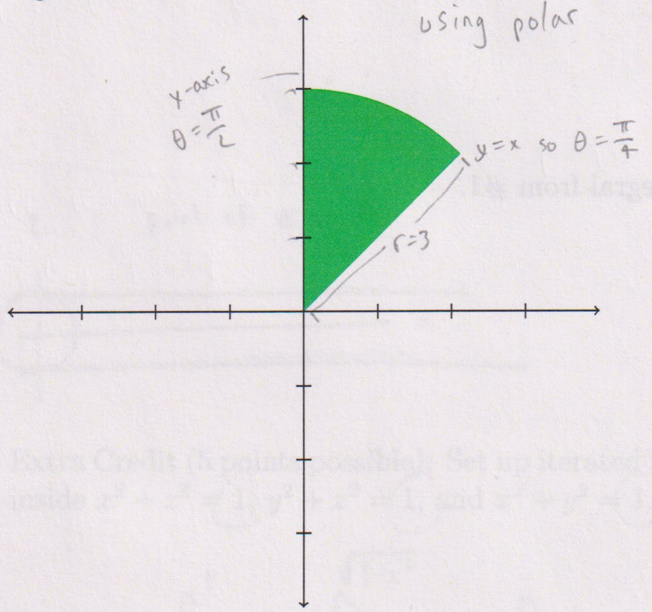


$$\int_0^3 \int_{y=x}^{y=3} f(x,y)\,dy\,dx$$

Good

4. Set up limits of integration for finding the volume under $g(x,y) = 10 - x$ within the region shown below:

using polar



$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^3 \overbrace{(10 - r\cos\theta)}^{10-x \text{ but in polar}}\, r\,dr\,d\theta$$

Great!

5. Evaluate $\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} 3\,dz\,dy\,dx$

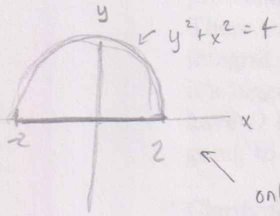
Full x

half y range

$z^2 = 4 - x^2 - y^2$

$z^2 + x^2 + y^2 = 4$

$V = \frac{4\pi}{3}(r)^3$



Sphere of radius 2

only top hemisphere

only positive y section

1/4 of total sphere

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} 3\,dz\,dy\,dx = 3 \cdot \frac{1}{4} \cdot \frac{4\pi}{3}(2)^3 = \boxed{8\pi}$$

"density"
(constant pulled out)

Excellent

6. Show that the Jacobian for the conversion from rectangular to polar coordinates is what it is.

Welp. ok.

we know that $x = r\cos\theta$ and $y = r\sin\theta$

The Jacobian is defined to be this bit

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} \cdot dr\,d\theta$$

If you were doing an integral, you'd need the $dr\,d\theta$ at the end.

So,

$$J = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta - (-r\sin^2\theta)$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta)$$

Excellent

$$= r \cdot 1$$

Thus, r is the Jacobian for the conversion from rectangular to polar. ■

7. Muffy is a calculus student at E.S.U., and she's having trouble with multiple integrals. Muffy says "Ohmygod, I so totally failed my Calc exam. There were totally impossible problems on it, and I think it's totally bad, and my daddy is going to sue the school. There was this one that was like, you were supposed to find the region-thingy for this integral $\iiint_D 4 - x^2 - y^2\, dV$ to be the biggest it could be, and I said, like, obviously it's bigger if you do it for a bigger regionthingy, right? So, it must be biggest if you have D be like all negative infinity to infinity, right? But I got no points, so Daddy's going to get the professor fired."

   Clarify for Muffy, in terms she can understand, how she should think about a problem like this, and what region D in fact maximizes the given integral.

So Muffy, first of all, even if one exam question has a problem, I'm not sure that's a good reason for firing someone. It's not actually "impossible", so you're still being tested on how much you understand.

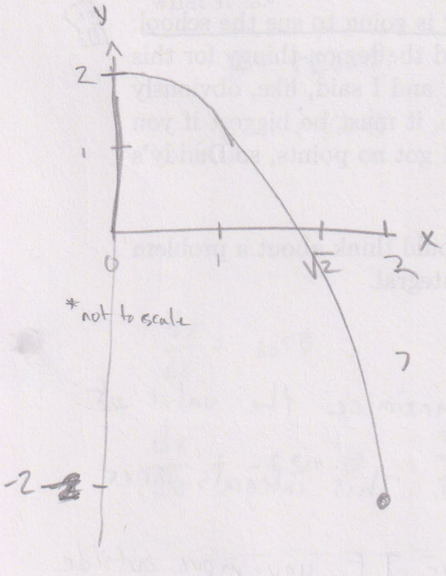
Second of all, I think you might have gotten the problem a little bit wrong. Are you sure it wasn't $\iint_D (4-x^2-y^2)\,dA$? I think that would make more sense. If it is, then notice that the integrand is positive (or the height of $z = 4-x^2-y^2$ is positive) inside the circle $x^2+y^2=4$, so a circle with radius 2 centered at the origin.

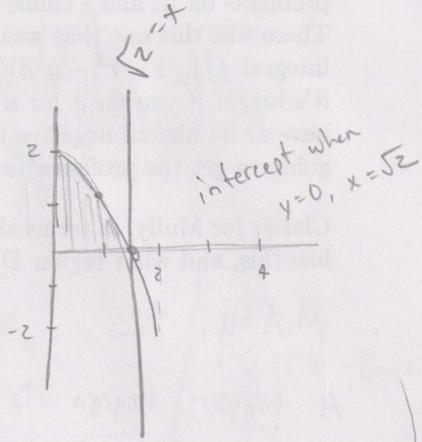
So just like if you integrate $f(x) = 4-x^2$ from -2 to 2 you get a positive number, but if you use a bigger interval the value of the integral actually gets smaller, the same thing happens here. If you integrate inside $x^2+y^2=4$ then each box in a Riemann sum is adding something to the total. Outside that domain you actually make the result smaller.

So back to the triple integral, now you'd get a positive density when $x^2+y^2 \le 4$ no matter what z is, so letting z go from negative infinity would make the value of the integral infinite. Even just part of $x^2+y^2 \le 4$ for z in $(-\infty, \infty)$ still gets an infinite result. It's not a meaningful question. I'm pretty sure something got miscopied.

The real answer is that $\iint_D (4-x^2-y^2)\,dA$ is maximized for the region D inside $x^2+y^2=4$.

8. Evaluate
$$\int_0^{\sqrt{2}} \int_0^{2-x^2} xe^{(x^2)}\ dy\,dx$$





$y = 2 - x^2$

$x = \sqrt{2-y}$

switching bounds Yes!

$$\int_0^2 \int_0^{\sqrt{2-y}} x e^{x^2}\, dx\, dy \qquad u = x^2 \quad du = 2x\,dx \quad \frac{du}{2} = x\,dx$$

$$\frac{1}{2} \int_0^2 \int_{u(0)}^{u(\sqrt{2-y})} e^u\, du\, dy = \frac{1}{2} \int_0^2 e^{x^2} \Big|_0^{\sqrt{2-y}} dy = \frac{1}{2} \int_0^2 e^{2-y} - e^0\, dy$$

$e^0 = 1$

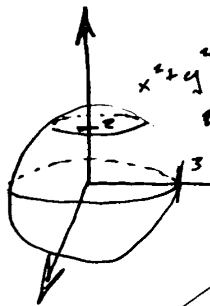
$$\frac{1}{2}\left(-e^{2-y} - y\right)\Big|_0^2 = \frac{1}{2}\left(-e^{\overset{0}{2-2}} - 2\right) - \frac{1}{2}\left(-e^{2-0} - 0\right)$$

$\frac{1}{2}(-1-2)$

Excellent!

$$\frac{-3}{2} + \frac{e^2}{2} \qquad \boxed{\frac{e^2-3}{2}}$$

9. Set up iterated integral(s) for the z coordinate of the centroid of the region inside a **sphere** with radius 3 centered at the origin but above $z = 0$ and below $z = 2$.



$x^2+y^2+z^2=3^2$

$z=\sqrt{9-r^2}$

$r=\sqrt{9-z^2}$

Cylindrical!

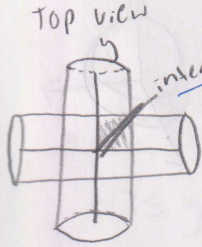
$$\bar{z} = \frac{\int_0^{2\pi}\int_0^{2}\int_0^{\sqrt{9-z^2}} z\cdot r\,dr\,dz\,d\theta}{\int_0^{2\pi}\int_0^{2}\int_0^{\sqrt{9-z^2}} 1\cdot r\,dr\,dz\,d\theta} = \underbrace{\frac{14\pi}{\frac{46\pi}{3}} = \frac{21}{23}}_{\text{Checking if it's reasonable}}$$

:)

$z = \pm\sqrt{1-y^2}$ $\qquad z = \pm\sqrt{1-x^2}$

10. Set up iterated integral(s) for the volume of the region inside both $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$.

Top view



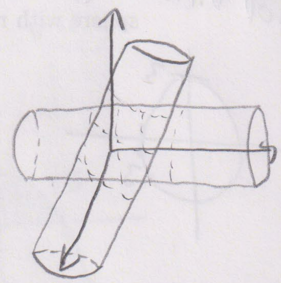
intersection

$\sqrt{1-x^2} = \sqrt{1-y^2}$

$1-x^2 = 1-y^2$

$x^2 = y^2$

$x = \pm y$



Yes!

$$8\int_0^1\int_0^x\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\,dz\,dy\,dx$$

well done!