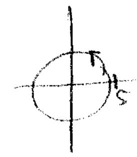
Exam 3 **Calc 3** **11/21/25**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. Give a parametrization and bounds for t to produce a counterclockwise circle with radius 5 centered at the origin.



$\vec{r}\,(t) = \langle 5\cos t,\ 5\sin t \rangle$

$0 \leq t \leq 2\pi$

Good

2. Is the vector field $\mathbf{F}(x, y) = \langle xy^2, x + y^2 \rangle$ conservative? How can you be sure?

$f_x = xy^2 \rightarrow f_{xy} = 2xy$

$f_y = x + y^2 \quad f_{yx} = 1$

not the same

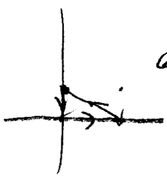
Because Clairaut's theorem says that functions with second order mixed partials that are continuous are equal, and the fact that the mixed partials of f are not the same, the vector field is not conservative.

Great!

P Q

3. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = 5x^2\mathbf{i} + 8x\mathbf{j}$ and C is a path composed of a line segment from $(0,0)$ to $(3,0)$ followed by a line segment from $(3,0)$ to $(0,1)$ and then a line segment from $(0,1)$ back to $(0,0)$.

This looks like a job for Green's theorem.



a closed, positively-oriented.

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_A 8-0 \, dA = 8 \cdot \text{area of the triangle}$$

$$8 \cdot \left(\frac{1}{2}\cdot 3 \cdot 1\right) = \boxed{12}$$

Excellent!

4. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = 2xy^3\mathbf{i} + 3x^2y^2\mathbf{j}$ and C is a line segment from $(1, 1)$ to $(3, 2)$.

Potential function

$f(x,y) = x^2 y^3$

F.T.L.I.

$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

$f(3,2) - f(1,1) \Rightarrow (3)^2 \cdot (2)^3 - (1)^2 \cdot (1)^3$

$9 \cdot 8 - 1$

$= 71$

Good

5. Let $\mathbf{F}$ be the vector field $\mathbf{F}(x, y, z) = 2x\mathbf{i} + x^3z^5\mathbf{j} + 3z\mathbf{k}$. Let S be the sphere with radius 3, centered at the origin and oriented outward. Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

Because S is a closed surface, use Divergence Theorem

$$\text{div}(\vec{F}) = \nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle 2x, x^3z^5, 3z \rangle = 2+0+3$$

$$= \boxed{5}$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div}\,\vec{F}\, dV$$

$$\hookrightarrow \iiint_E 5\, dV = 5 \iiint_E 1\, dV$$

$\hookrightarrow$ vol of sphere w/ $r=3$

$$= 5\left[\frac{4}{3}\pi(3)^3\right] = \boxed{180\pi}$$

Excellent

6. Show that for any vector field $\mathbf{F}(x, y, z)$ whose component functions have continuous partial derivatives, div(curl $\mathbf{F}$) $= 0$. Make it clear how the requirement that the partials be continuous is important.

let $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \left\langle \frac{\partial}{\partial y}R - \frac{\partial}{\partial z}Q, -\left(\frac{\partial}{\partial x}R - \frac{\partial}{\partial z}P\right), \frac{\partial}{\partial x}Q - \frac{\partial}{\partial y}P \right\rangle$$

$$\text{div}(\text{curl } F) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \text{curl } F =$$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}R - \frac{\partial}{\partial x}\frac{\partial}{\partial z}Q + \frac{\partial}{\partial y}\frac{\partial}{\partial z}P - \frac{\partial}{\partial y}\frac{\partial}{\partial x}R + \frac{\partial}{\partial z}\frac{\partial}{\partial x}Q - \frac{\partial}{\partial z}\frac{\partial}{\partial y}P =$$

Because the components have continuous partial derivitive's Clairout's Theorem applies which states $f_{xy} = f_{yx}$. This can be restated so that $\frac{\partial}{\partial x}\frac{\partial}{\partial y}R = \frac{\partial}{\partial y}\frac{\partial}{\partial x}R$, $\frac{\partial}{\partial x}\frac{\partial}{\partial z}Q = \frac{\partial}{\partial z}\frac{\partial}{\partial x}Q$, and $\frac{\partial}{\partial y}\frac{\partial}{\partial z}P = \frac{\partial}{\partial z}\frac{\partial}{\partial y}P$

Excellent!

So

$$\underbrace{\frac{\partial}{\partial x}\frac{\partial}{\partial y}R - \frac{\partial}{\partial y}\frac{\partial}{\partial x}R}_{0} + \underbrace{\frac{\partial}{\partial z}\frac{\partial}{\partial x}Q - \frac{\partial}{\partial x}\frac{\partial}{\partial z}Q}_{0} + \underbrace{\frac{\partial}{\partial y}\frac{\partial}{\partial x}P - \frac{\partial}{\partial x}\frac{\partial}{\partial y}P}_{0}$$

$$0 + 0 + 0 = 0$$

thus $\text{div}(\text{curl } \vec{F}) = 0$

for all fields where component functions have continuous partial derivitaties

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Crap. This line integral stuff is crazy. There's all these different ways, but I figure the shortest is best, right? So they said something about a time it's really simple, like, if the curl of the vector field is zero then line integrals on closed paths are always zero too. Is that true?"

Help Biff by explaining as clearly as you can whether what he heard is true, and how you know.

This is because of Stokes' Theorem, which states

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\, \vec{F} \cdot d\vec{S}$$

So if $\text{curl}\, \vec{F} = 0$, then $\iint_S \text{curl}\, \vec{F}\, d\vec{S}$ must evaluate to zero as well. This means, that when $\text{curl}\, \vec{F} = 0$, $\int_C \vec{F} \cdot d\vec{r} = 0 = \iint_S \text{curl}\, \vec{F} \cdot d\vec{S}$

Thus, when a vector field's curl is zero, the line integral on a closed path is zero. ~~The closed~~ p ~~A lin~~

In order for Stoke's Theorem to apply, the line integral must be a closed path. Then, as above, it evaluates to 0.

Excellent!

8. Let $\mathbf{F}$ be the vector field $\mathbf{F}(x,y,z) = \langle -y, x, 5\rangle$. Let S be the cylinder $x^2+y^2=4$ between $z=0$ and $z=5$. Find $\iint_S \mathbf{F}\cdot d\mathbf{S}$.

The Looong Way

I. $\vec{r}(u,v) = \langle 2\cos u, 2\sin u, v\rangle \qquad \begin{matrix} 0 \le u \le 2\pi \\ 0 \le v \le 5 \end{matrix}$

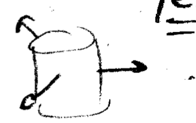
II. $\vec{F}(\vec{r}) = \langle -2\sin u, 2\cos u, 5\rangle$

III $\vec{r}_u \times \vec{r}_v \qquad r_u = \langle -2\sin u, 2\cos u, 0\rangle$

$r_v = \langle 0, 0, 1\rangle$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin u & 2\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 2\cos u, 2\sin u, 0\rangle$$

oriented out? YES ✓

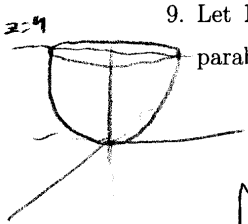


IV $\displaystyle\int_0^5\int_0^{2\pi} \langle -2\sin u, 2\cos u, 5\rangle\cdot\langle 2\cos u, 2\sin u, 0\rangle\, du\,dv$

V Evaluate $\displaystyle\int_0^5\int_0^{2\pi} \cancel{-4\sin u\cos u} + \cancel{4\cos u\sin u} + 0\, du\,dv$

$$\int_0^5\int_0^{2\pi} 0\, du\,dv = \boxed{0}$$

Excellent!



9. Let $\mathbf{F}$ be the vector field $\mathbf{F}(x, y, z) = \langle xz, y^3, -x^3 \rangle$. Let S be the portion of the paraboloid $z = x^2 + y^2$ below $z = 4$ with upward orientation. Find $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

Curl? Stokes? I'm stoked!

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$$

Let C be the circle $x^2+y^2=4$ at $z=4$

I. $\vec{r}(t) = \langle 2\cos t, 2\sin t, 4 \rangle$ $\quad 0 \le t \le 2\pi$ going around correctly? YES palm on right side

II $\vec{F}(\vec{r}) = \langle 8\cos t, 8\sin^3 t, -8\cos^3 t \rangle$

III $\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 0 \rangle$ oriented correctly? YES

IV $$\oint_0^{2\pi} \langle 8\cos t, 8\sin^3 t, -8\cos^3 t \rangle \cdot \langle -2\sin t, 2\cos t, 0 \rangle \, dt$$

V Evaluate

Excellent!

$$\oint_0^{2\pi} -16\cos t \sin t + 16\sin^3 t \cos t \, dt$$

W

$$-16 \oint_0^{2\pi} \cos t \sin t \left(1 + \sin^2 t\right) dt$$

one revolution of these trig fxns equals 0!

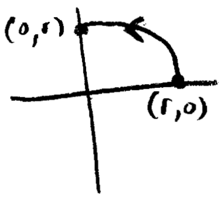
$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \boxed{0}$$

$\nabla x \cdot xy = \langle -y,$ $\qquad \frac{\partial}{\partial y} -xy = -x$

10. Let $\mathbf{F}(x, y) = \langle -y, x\rangle$, and let C be the first quadrant portion of a circle with some radius r centered at the origin, traversed counterclockwise from $(r, 0)$ to $(0, r)$. For what radius r will $\int_C \mathbf{F} \cdot d\mathbf{r} = 5$?

$\int -y\,dr = -yx + c$

$\int x\,dy = xy + c$



$f_x = -y \rightarrow f_{xy} = -1$

$f_y = x \rightarrow f_{yx} = 1$

$>$ not same so not cons. use normal way

$\vec{r}(t) = \langle r\cos t, r\sin t\rangle \quad 0 \le t \le \pi/2$

r is constant for radius

$\vec{F}(\vec{r}(t)) = \langle -r\sin t, r\cos t\rangle$

$\vec{r}\,'(t) = \langle -r\sin t, r\cos t\rangle$

$$\int_0^{\pi/2} \langle -r\sin t, r\cos t\rangle \cdot \langle -r\sin t, r\cos t\rangle \, dt$$

$$= \int_0^{\pi/2} r^2\sin^2 t + r^2\cos^2 t \, dt$$

$$= \int_0^{\pi/2} r^2(\sin^2 t + \cos^2 t)\, dt \rightarrow \int_0^{\pi/2} r^2 \, dt = r^2 t\Big|_0^{\pi/2} = \frac{r^2\pi}{2}$$

$$\frac{r^2\pi}{2} = 5$$

$$r^2\pi = 10$$

$$r^2 = \frac{10}{\pi}$$

$$\boxed{r = \sqrt{\frac{10}{\pi}}}$$

Excellent!