Calculus IV   Exam 3   Fall 1999   12/2/99

Each problem is worth 10 points. Be sure to show all work for full credit. Please circle all answers and keep your work as legible as possible. Not liable for consequences of acts of God.

1. Compute $\int_C (\overset{P}{y\mathbf{i}} - \overset{Q}{x\mathbf{j}})\cdot d\mathbf{r}$ where the path C is a line segment from (1,3) to (2,0).

10

Line Integral → Not a closed curve. Is there a potential function

$\frac{\partial Q}{\partial x} = -1$ $\frac{\partial P}{\partial y} = 1$ $\frac{\partial Q}{\partial x} \neq \frac{\partial P}{\partial y}$ ∴ No Potential function.

∴ Long way. $\vec{r}(t) = \langle 1+t,\ 3-3t \rangle$

$x(t) = 1+t$ $0 \leq t \leq 1$ $\vec{F}(\vec{r}(t)) = \langle (3-3t),\ -(1+t) \rangle$

$y(t) = 3-3t$ $\vec{r}\,'(t) = \langle 1, -3 \rangle$

$$\int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \int_0^1 \langle (3-3t), (-1-t) \rangle \cdot \langle 1, -3 \rangle dt$$

$$= \int_0^1 (3-3t - 3(-1-t))\, dt = \int_0^1 (3-3t+3+3t)dt = \int_0^1 6\, dt$$

$$= 6\left[t\right]_0^1 = \boxed{6}$$

Beautiful!

10

2. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ for $\mathbf{F}(x,y,z)=y\mathbf{i}+(x-z)\mathbf{j}+(2-y)\mathbf{k}$ for any path C from (0,$\overset{a}{3}$,-2) to ($\overset{b}{2}$,0,1). How do you know the answer is independent of the particular path chosen?

$f = xy - yz + 2z$ (potential function, ∴ F is conservative)
use Fund. Thm

$\int_C F\cdot dr = f(\vec{r}(b)) - f(\vec{r}(a))$

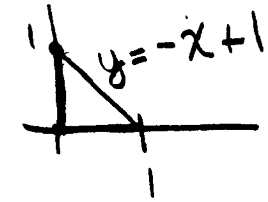
$= f(2,0,1) - f(0,3,-2) = 2 - (+6-4) = \boxed{0}$

the answer is independent of path because the vector field is conservative and the path is of no consequence.

Great

3. Evaluate $\int_C \overset{P}{x^4}\,dx + \overset{Q}{xy}\,dy$, where C is the triangular curve consisting of the line segments from (0,0) to (1,0), from (1,0) to (0,1), and from (0,1) to (0,0).

This is a continuous closed line integral, so Greens theorem is the way to go.



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$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dA$$

Good

$$\int_0^1 \int_0^{-x+1} y - 0 \quad dy\,dx$$

$$\int_0^1 \int_0^{-x+1} y\,dy\,dx = \int_0^1 \left[\frac{y^2}{2}\right]_{y=0}^{-x+1} dx = \frac{1}{2}\int_0^1 x^2 - 2x + 1\,dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - x^2 + x\right]_{x=0}^{1} = \frac{1}{2}\left[\frac{1}{3} - 1 + 1\right] = \boxed{\frac{1}{6}}$$

4. A really goofy cult leader has predicted that at midnight on New Year's Eve 1999, every point within and around the Earth will suddenly begin radiating jelly beans according to the vector field $\mathbf{F}(x,y,z)=\underset{P}{x^3}\mathbf{i}+\underset{Q}{y^3}\mathbf{j}+\underset{R}{z^3}\mathbf{k}$. Just in case this incredibly unlikely event should come to pass, compute the total number of jelly beans that would radiate outward through the surface of the Earth, i.e. the flux integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$ where S is a sphere of radius approximately 4000 miles.

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Since sphere it is a closed surface + we can use divergence theorem

$$\operatorname{div} F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$= 3x^2 + 3y^2 + 3z^2$$

← need to switch to spherical coordinates

$x^2+y^2+z^2=\rho^2$ so $3x^2+3y^2+3z^2=3\rho^2$

Great

$$\iiint_E \operatorname{div} F \, dV$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{4000} 3\rho^2\rho^2\sin\phi \, d\rho\, d\phi\, d\theta = \iiint 3\rho^4 \sin\phi \, dV$$

$$\int_0^{2\pi}\int_0^{\pi} \sin\phi \; \tfrac{3}{5}\rho^5\Big|_0^{4000} d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi} 6.144E^{17}\sin\phi \, d\phi\, d\theta \rightarrow$$

$$\int_0^{2\pi} 6.144E^{17} -\cos\phi\Big|_0^{\pi} d\theta$$

$$\int_0^{2\pi} 6.144E^{17}(1 - -1)\, d\theta = \int_0^{2\pi} 1.2288E^{18}\, d\theta = 2\pi(1.2288E^{18})$$

* a lot!

5. Compute the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$ where **F(x,y,z)=yi-xj-2k** and **S** is the surface of the paraboloid $z=x^2+y^2$ (with upward orientation) within the cylinder $x^2+y^2=4$

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$\boxed{\text{14.7 Long Way}}$

$x(u,v) = u$

$y(u,v) = v$

$z(u,v) = u^2+v^2$

$r(u,v) = \langle u, v, u^2+v^2 \rangle$

$F(r(r)) = \langle v, -u, -2 \rangle$

$\vec{r}_u = \langle 1, 0, 2u \rangle$

$\vec{r}_v = \langle 0, 1, 2v \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 1, 0, 2u \rangle \times \langle 0, 1, 2v \rangle$

$= 1\hat{k} - 2v\hat{j} - 2u\hat{i}$

$= \langle -2u, -2v, 1 \rangle$

Excellent

$\iint \langle -2u, -2v, 1 \rangle \cdot \langle v, -u, -2 \rangle \, dA$

$\iint (-2uv + 2uv - 2) \, dA$

$\int_0^{2\pi}\int_0^2 (-2r) \, dr \, d\theta$

$\int [-r^2]_0^2 \, d\theta = \int_0^{2\pi} (-4) \, d\theta$

$= [-4\theta]_0^{2\pi}$

$= -8\pi$

Use Stokes' Theorem to compute the surface integral $\iint_S \text{curl } \mathbf{F}\cdot d\mathbf{S}$ where

$= -2(\pi R^2)$
$-2(\pi(4)) = -8\pi$

6. ***Use Stokes' Theorem*** **to compute the surface integral** $\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$ **where F(x,y,z)=2yi+j+xyk and S is the surface of the paraboloid** $z=x^2+y^2$ **(with upward orientation) within the cylinder** $x^2+y^2=4$.

10

$\iint_S \text{curl } F\cdot ds = \int_C F\cdot dr$

I. $x(t) = 2\cos t$
$y(t) = 2\sin t$
$z(t) = 4$

$\vec{r}(t) = \langle 2\cos t, 2\sin t, 4\rangle$

II. $F(\vec{r}(t)) = \langle 4\sin t, 1, 4\cos t\sin t\rangle$

Great

III. $\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 0\rangle$

IV. $\int_C F\cdot dr = \int_C -8\sin^2 t + 2\cos t + 0$

$\int_C F\cdot dr = \int_0^{2\pi} -8\sin^2 t\,dt + \int_0^{2\pi} 2\cos t\,dt$

$= -8\int \sin^2 t\,dt + \left[2\sin t\right]_0^{2\pi} \rightarrow 2\sin 2\pi - 2\sin 0 = 0] \rightarrow$ 0 ...

7. Show that for any function f(x,y,z) with continuous second partials, curl(grad f) = **0**. Make clear how you use the continuity condition.

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Let $f(x,y,z)$ be a function s.t. $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial z^2}$ exists

so $\nabla f(x,y,z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$. we take curl $(\nabla f) = \nabla \times \nabla f$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \vec{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial z} \end{vmatrix} \vec{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{vmatrix} \vec{k}$$

$$= \left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)\right)\vec{i} - \left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right)\right)\vec{j} + \left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right)\vec{k}$$

$$= \left(\frac{\partial^2 f}{\partial^2 yz} - \frac{\partial^2 f}{\partial^2 zy}\right)\vec{i} - \left(\frac{\partial^2 f}{\partial^2 xz} - \frac{\partial^2 f}{\partial zx}\right)\vec{j} + \left(\frac{\partial^2 f}{\partial xy} - \frac{\partial^2 f}{\partial yx}\right)\vec{k}$$

$= 0\vec{i} - 0\vec{j} + 0k$: (by Clairaut's Thm)

$= \langle 0, 0, 0 \rangle = \vec{0}$

Well done

8. Show that Green's Theorem is a special case of Stokes' Theorem (where the surface lies entirely in the xy-plane) by applying Stokes' Theorem to **F**(x,y,z)=P(x,y)**i**+Q(x,y)**j**+0**k**.

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$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\ \vec{F}\cdot d S$; curl $\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & 0 \end{vmatrix}\vec{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & 0 \end{vmatrix}\vec{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix}\vec{k}$

$= \left(0 - \frac{\partial Q}{\partial z}\right)\vec{i} - \left(0 - \frac{\partial P}{\partial z}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$

but $\frac{\partial Q}{\partial z} = \frac{\partial P}{\partial z} = 0$ so curl $\vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$

$= \iint_S \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \cdot dS$ but since $z=0$ we have region D and an associate not a surfa

so we get $\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$

9. Buffy says "Like, Calculus is *so* unfair. My professor gave us this test, and we were supposed to show that for this line integral thingy it didn't matter what way you went, like the path or whatever. So I like, did it for two different parama-whatevers, and, like, I got the same thing both ways, which is like a miracle for me anyway, but my professor wrote all this bad stuff that I totally don't understand about how that wasn't the right thing to do. She gave me almost no credit at all! But, my *God*, how else could I do it? I mean, it's not like I have enough time to do every way of connecting those two points, cause there's, like, *lots* of them."

10

Explain (clearly enough for Buffy to understand) how such a thing can be done without actually trying an infinite number of paths, and tell her what's flawed about her approach.

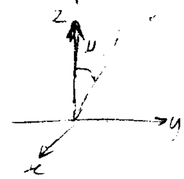
Well Buffy, it all depends on whether your vector field has a potential function, that is, whether it is the gradient of some function. You can determine this easily enough by taking the curl of your vector field. If this curl comes out to zero, than your field is the gradient of some function, and line integrals through that field are idependent of path. It's called a conservative vector field. However, if this curl is not 0, then your field is not a gradient and has no potential function. It wouldn't be too difficult for your professor to concoct some strange vector field which is not conservative, but through which line integrals with two or more different paths would yield the same result. This would only be a coincidence, and could have led you astray using the method you tried on your test. If indeed a vector field has a potential function (is conservative), then line integrals between the same two points will always yield the same result, as long as the paths are "reasonably well-behaved."

Well put

10. Show that the surface area of a unit sphere (with parametrization x(u,v)=sin u cos v, y(u,v)=sin u sin v, z(u,v)=cos u) is 4π.

In case anything goes wrong :)

Surface area $= \frac{d}{dr}$ Volume $= \frac{d}{dr}\frac{4}{3}\pi r^3 = \frac{4}{3}\cdot 3\pi r^2 = 4\pi r^2$ r is given as 1 ∴ Surface area $= \underline{4\pi}$



$0 \le u \le \pi$

$0 \le v \le 2\pi$

$r = \langle x\ y\ z \rangle = \langle \sin u \cos v, \sin u \sin v, \cos u \rangle$

$r_u = \langle \cos u \cos v, \cos u \sin v, -\sin u \rangle$

$r_v = \langle -\sin u \sin v, \sin u \cos v, 0 \rangle$

W

$r_u \times r_v = \begin{vmatrix} i & j & k \\ \cos u \cos v & \cos u \sin v & -\sin u \\ -\sin u \sin v & \sin u \cos v & 0 \end{vmatrix} = i(0 + \sin^2 u \cos v) + j(\sin^2 u \sin v - 0) + k(\sin u \cos^2 v \cos u + \cos u \sin^2 v \sin u)$

$r_u \times r_v = i(\sin^2 u \cos v) + j(\sin^2 u \sin v) + k(\sin u \cos u)$

$|r_u \times r_v| = [\sin^4 u \cos^2 v + \sin^4 u \sin^2 v + \sin^2 u \cos^2 u]^{\frac{1}{2}}$

$= [\sin^4 u [\cos^2 v + \sin^2 v] + \sin^2 u \cos^2 u]^{\frac{1}{2}} = [\sin^4 u + \sin^2 u \cos^2 u]^{\frac{1}{2}}$

$= [\sin^2 u (\sin^2 u + \cos^2 u)]^{\frac{1}{2}} = [\sin^2 u]^{\frac{1}{2}} = \underline{\sin u}$

$A(S) = \iint_D |r_u \times r_v|\, dA = \iint_D \sin u\, dA = \int_0^{2\pi}\int_0^{\pi} \sin u\, du\, dv$

$= \int_0^{2\pi} -\cos u \Big|_0^{\pi} dv = 1 - (-1) \int_0^{2\pi} 1\, dv$

$2 \times 2\pi = \underline{\underline{4\pi}}$ lucky me

Beautiful

Extra Credit (5 points possible):

+2

You might think the surface area of the ellipsoid with parametrization x(u,v)=2 sin u cos v, y(u,v)=2 sin u sin v, z(u,v)=cos u would work out very much like that of the sphere in problem 10, but it turns out to be much harder. See what you can do with it (estimates might be good).

Extra Credit (5 points possible):

+5

You might think the surface area of the ellipsoid with parametrization x(u,v)=2 sin u cos v, y(u,v)=2 sin u sin v, z(u,v)=cos u would work out very much like that of the sphere in problem 10, but it turns out to be much harder. See what you can do with it (estimates might be good).

$r(u,v) = \langle 2\sin u \cos v,\ 2\sin u \sin v,\ \cos u \rangle$

$r_u = \langle 2\cos v \cos u,\ 2\cos u \sin v,\ -\sin u \rangle$

$r_v = \langle -2\sin u \sin v,\ 2\sin u \cos v,\ 0 \rangle$

(on back)

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ 2\cos v \cos u & 2\cos u \sin v & -\sin u \\ -2\sin u \sin v & 2\sin u \cos v & 0 \end{vmatrix} =$$

$\int \sec t\,dt = \ln|\sec u + \tan u|$

$\ln|\sec(\tan^{-1}(-\sqrt{3})) + \tan(\tan^{-1}(-\sqrt{3}))|$

$-\ln|$

Wow

$$\hat{i}(0 + 2\sin^2 u \cos v) + (0 + 2\sin^2 u \sin v)\hat{j}$$

$$-(4\cos^2 v \sin u \cos u + 4\sin^2 v \sin u \cos u)\hat{k}$$

$$= \langle 2\sin^2 u \cos v,\ 2\sin^2 u \sin v,\ 4\sin u \cos u \rangle$$

$$|r_u \times r_v| = \sqrt{4\sin^4 u \cos^2 v + 4\sin^4 u \sin^2 v + 16\sin^2 u \cos^2 u}$$

$$= \sqrt{4\sin^4 u + 16\sin^2 u \cos^2 u} = 2\sin u\sqrt{\sin^2 u + 4\cos^2 u}$$

$$= 2\sin u\sqrt{3\cos^2 u + 1}$$

Let $x = \cos u$

$dx = -\sin u\,du$

$$\int_0^{2\pi}\int_0^{\pi} 2\sin u\sqrt{3\cos^2 u + 1}\,du\,dv =$$

Let $x = \frac{\tan t}{\sqrt{3}}$

$dx = \frac{1}{\sqrt{3}} \cdot \sec^2 t\,dt$

$$= -2\int_0^{2\pi}\int_1^{-1}\sqrt{3x^2+1}\,dx\,dv$$

$$-\frac{2}{\sqrt{3}}\int_0^{2\pi}\int_{\tan^{-1}(\sqrt{3})}^{\tan^{-1}(-\sqrt{3})}\sec^3 t\,dt\,dv$$

$\int \sec^3 t\,dt$

$\frac{d\sec t}{dt} = \sec t \tan t$

Let $u = \sec t$ $du = \sec t \tan t\,dt$

$dv = \sec^2 t\,dt$ $v = \tan t$

$$-\frac{2}{\sqrt{3}}\int_0^{2\pi}[-5.781]\,dv = -\frac{2}{\sqrt{3}} \cdot -10.13 \cdot 2\pi = 34.69$$

$\sec t \tan t - \int \tan^2 t \sec t\,dt$

$-4\sqrt{3} = \int \sec^3 t\,dt + \int \sec t\,dt$

$\Rightarrow \int \sec^3 t\,dt = -2\sqrt{3} + \frac{1}{2}\int \sec t\,dt$

By calculator $\Rightarrow \approx \boxed{34.69}$

$= -2\sqrt{3} - \frac{1}{2}\left(\ln\left|\frac{2+\sqrt{3}}{2-\sqrt{3}}\right|\right.$

Directly $\Rightarrow 2\pi\left[4 + \frac{1}{\sqrt{3}}\ln\left|\frac{2+\sqrt{3}}{2-\sqrt{3}}\right|\right]$