Calculus III Exam 1 Spring 2000 2/4/2000

Each problem is worth 10 points. Be sure to show all work and justifications for full credit. Please circle all answers and keep your work as legible as possible. No animals were used in the making of this exam.

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1. Find the limit of the sequence $\left\{\dfrac{n^2}{2n^2-3n}\right\}$, giving justifications for important steps in your reasoning.

Theorem [illegible]

$$\lim_{n\to\infty} \frac{n^2}{2n^2-3n} = \lim_{n\to\infty} \frac{1}{2-\frac{3}{n}} = \boxed{\frac{1}{2}}$$

Algebra, divide by highest power of n.

goes to 0

Great

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2. Find the first four partial sums of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

– You must add the previous sum to the new term to get the partial sum.

$s_1 = \frac{1}{1^2} = 1$

$s_2 = 1 + \frac{1}{2^2} = \frac{5}{4}$

$s_3 = \frac{5}{4} + \frac{1}{3^2} = \frac{5}{4} + \frac{1}{9} = \frac{45+4}{36} = \frac{49}{36}$

$s_4 = \frac{49}{36} + \frac{1}{4^2} = \frac{49}{36} + \frac{1}{16}$

$\frac{49(16)+36}{36(16)} = \frac{784+36}{576} = \frac{820}{576} = \frac{410}{288} = \frac{205}{144}$

36, 16, 216, 36, 49, 294, 490, 784, 36

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3. Give an example of a sequence a_n which converges, but for which $\sum_{n=1}^{\infty} a_n$ diverges.

When $a_n = \frac{1}{n}$, the sequence converges (because $\lim_{n\to\infty} \frac{1}{n} = 0$), but $\sum_{n=1}^{\infty} a_n$ diverges. (because in the p-series where $a_n = \frac{1}{n^p}$, $p \le 1$ and therefor must diverge)

Good

4. Find the sum of the series $\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\ldots+\frac{2}{3^n}+\ldots$

$\sum_{n=1}^{\infty} \frac{2}{3^n}$ $\qquad \frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\ldots$

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geometric series

$a = 2/3 \quad r = 1/3 \quad |r| < 1 \Rightarrow$ converges

$$\frac{a}{1-r} \Rightarrow \frac{2/3}{1-1/3} = \frac{2/3}{2/3} = 1$$

5. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}}$ converges or diverges.

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$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\cancel{x}}{\cancel{x}\sqrt{1+\frac{1}{n}}} = \frac{1}{\sqrt{1+\frac{1}{\infty}}} = 1 > 0$$

But we know that $\sum \frac{1}{n}$ diverges - harmonic, therefore $\frac{1}{\sqrt{n^2+n}}$ also $\boxed{\text{diverges}}$ by the Limit Comparison Test

Great

6. Use the Ratio Test to show whether the series $\sum_{n=1}^{\infty} \frac{n+1}{n!}$ converges or diverges.

$a_n = \frac{n+1}{n!}$

$a_{n+1} = \frac{(n+1)+1}{(n+1)!}$ $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \frac{\lim_{n\to\infty} \frac{n+2}{(n+1)!}}{\lim_{n\to\infty} \frac{n+1}{n!}} = \lim_{n\to\infty} \frac{n+2}{(n+1)!} \bullet \frac{n!}{n+1} = \lim_{n\to\infty} \frac{n+2}{(n+1)(n+1)} = \frac{\frac{n}{n^2}+\frac{2}{n^2}}{\frac{n^2}{n^2}+\frac{2n}{n^2}+\frac{1}{n^2}} \to \frac{0}{1}$

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$\left|\frac{a_{n+1}}{a_n}\right| = L < 1$ series converges

$\sum_{n=1}^{\infty} \frac{n+1}{n!}$ $L=0$ converges by Ratio Test

Excellent

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7. Determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n\cdot\ln n}$ converges or diverges.

Integral test

$\int^{\infty} f(x)\,dx$ We use the integral test & we treat this series as a function

$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{1}{x\ln x} = \int_{x=2}^{x=\infty} \frac{1}{u}\,du$ $u = \ln x$, $du = \frac{1}{x}\,dx$

$= \Big[\ln|u|\Big]_{x=2}^{x=t} = \ln|\ln x|\Big]_2^t$

$\ln|\ln t| - \ln|\ln 2|$

∞ − some small #

this diverges

Nice

8. Ken is a calculus student from California, and he's gotten a little confused. Ken says "Wow, I skipped my calc class a few times, and now when I started going back it's totally *whacked!* Man, the prof is talking about finding out what it adds up to when you, like, add these things together, and there's infinitely many of these things that we're supposed to add up. Isn't that completely whacked? I mean, if you add infinitely many things you *gotta* automatically get infinity, right?"

Explain *briefly* and *clearly* to Ken how we can talk about the sum of infinitely many things and how the sum need not always be infinite. Just naming theorems to him probably won't help -- you need to actually get the idea across to him.

LET US LOOK AT A PARTICULAR SERIES $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$

THE TERMS OF THE SEQUENCE ARE: $1, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64} \ldots$

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NOW LOOK AT THE PARTIAL SUMS (The sum of the first n many terms)

$s_1 = \frac{1}{2} \quad = \frac{1}{2}$

$s_2 = \frac{1}{2} + \frac{1}{4} \quad = 3/4$

$s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \quad = 7/8$

$s_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \quad = 15/16$

IN THIS SERIES, EACH NEW TERM IS $\frac{1}{2}$ THE VALUE OF THE PRECEDING TERM.

AS WE ADD THESE TERMS WE SEE THE PARTIAL SUMS INCREASE. AS WE ADD SMALLER AND SMALLER TERMS, THE SUMS BEGIN TO INCREASE LESS EACH TIME.

LOOK AT THE NEW SEQUENCE OF PARTIAL SUMS: $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots$

EACH TIME WE ADD $\frac{1}{2}$ THE value of the preceding term WHICH IS ALSO ALWAYS $\frac{1}{2}$ THE DISTANCE BETWEEN THE PARTIAL SUM AND 1. Good

$\therefore$ Because we CAN ONLY GET 1/2 THE WAY CLOSER TO 1 EACH TIME, THE SERIES OF $\sum_{n=1}^{\infty} \frac{1}{2^n}$ CANNOT EVER GET BIGGER THAN 1, NO MATTER HOW MANY NUMBERS WE ADD. Therefore, it is possible to add up infinitely many things and never reach infinity.

Excellent

9. Consider the series $1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\ldots$ which continues with terms of the form 1/n with signs alternating in pairs. What can you say about the convergence or divergence of this series?

rewrite series

$$\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}+\frac{1}{2n}\right)\cdot(-1)^{n-1}$$

let $\{b_n\} = \left\{\frac{1}{2n-1}+\frac{1}{2n}\right\}$

$\{b_n\} = 1+\frac{1}{2},\ \frac{1}{3}+\frac{1}{4},\ \frac{1}{5}+\frac{1}{6},\ \frac{1}{7}+\frac{1}{8},\ \ldots,\ b_n$

$b_1 > b_2 > b_3 > b_4 > \ldots b_n$

★ $b_{n+1} < b_n$ for all n

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take $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{2n-1}+\frac{1}{2n}$

$\Rightarrow \lim_{n\to\infty} \frac{1}{2n-1}\frac{(\frac{1}{n})}{(\frac{1}{n})} + \lim_{n\to\infty}\frac{1}{2n}\frac{(\frac{1}{n})}{(\frac{1}{n})}$

$\Rightarrow \lim_{n\to\infty}\frac{\frac{1}{n}}{2-\frac{1}{n}} + \lim_{n\to\infty}\frac{\frac{1}{n}}{2}$

$\Rightarrow 0 + 0$

$= \boxed{0}$

THEREFORE, by alternating series test, since

a) $b_{n+1} \leq b_n$ for all n

and b) $\lim_{n\to\infty}\{b_n\} = 0$

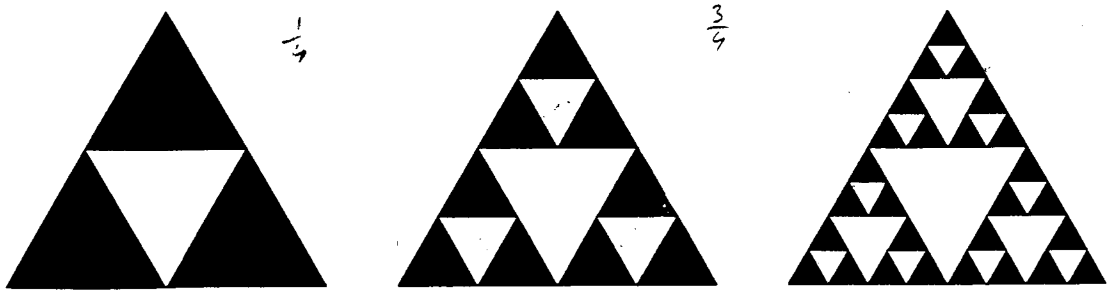
and $b_n > 0$

the SERIES $\boxed{\text{CONVERGES}}$

Excellent

10. Begin with an equilateral triangle with a total area of 1 and successively remove smaller triangles from it in the manner shown below:

$\frac{1}{4}$ $\frac{3}{4}$



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(a) If a_n is the *total* area removed in step n *alone*, find a_1, a_2, a_3, and a_4.

$a_1 = \frac{1}{4}$

$a_2 = \frac{3}{16}$

$a_3 = \frac{9}{64}$

$a_4 = \frac{27}{256}$

(b) If we continue the process indefinitely, express the total area removed as a series and find the sum of that series.

$$\sum_{n=1}^{\infty} \frac{1}{4}\left(\frac{3}{4}\right)^{n-1}$$

$\frac{1}{4}, \frac{3}{16}, \frac{9}{64}$

Geometric $\frac{A}{1-r}$ $r = \frac{3}{4}$ $A = \frac{1}{4}$

Great

$$\frac{\frac{1}{4}}{1-\frac{3}{4}} = 1$$

Extra Credit (5 points possible):

[1 pt.] Give an example of a sequence $\{a_n\}$ which converges to 6. $\{6\}$

[2 pts.] Give an example of sequences $\{a_n\}$ and $\{b_n\}$ such that $\{a_n + b_n\}$ converges to 6. $\{2\}$ $\{4\}$

[2 pts.] Give an example of *divergent* sequences $\{a_n\}$ and $\{b_n\}$ such that $\{a_n + b_n\}$ converges to 6.

$\{6\sin^2 n\}$ $\{6\cos^2 n\}$ Neaky, but it works.