Calculus III Exam 3 Spring 2000 4/21/2000

Each problem is worth 10 points. Be sure to show all work and justifications for full credit. Please circle all answers and keep your work as legible as possible. No real animals were harmed in the making of this exam.

1. Write parametric equations for a line orthogonal to the plane -2(x-3)+5(y-1)-(z+7)=0.

the vector orthogonal to the plane is $\langle -2,5,-1\rangle$
on the point $(3,1,-7)$

Nice

so $\langle x,y,z\rangle = \langle 3,1,-7\rangle + t\langle -2,5,-1\rangle$

$x = 3-2t$
$y = 1+5t$
$z = -7-t$

2. Write an equation for the sphere with center at (3,-5,2) and radius 7.

$(x-3)^2+(y+5)^2+(z-2)^2 = 7^2$
$(x-3)^2+(y+5)^2+(z-2)^2 = 49$

3. If a talking chihuahua is thrown off the roof of the Physical Sciences Building in such a way that its position t seconds after release is given by $r(t)=\langle 3t, 2t, -4.9t^2+15t+37\rangle$, give the dog's velocity and acceleration vectors at time t.

10

$r(t) = \langle 3t, 2t, -4.9t^2+15t+37$ position
$r'(t) = \langle 3, 2, -9.8t+15\rangle$ velocity
$r''(t) = \langle 0, 0, -9.8\rangle$ acceleration

4. Find the equation of the plane including the lines [Hint: notice the point they have in common!]
r = <0, 0, 0> + t<-3, 1, 2> and **r** = <0, 0, 0> + s<5, -2, 0>.

$$\text{normal } \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 1 & 2 \\ 5 & -2 & 0 \end{vmatrix} = (0+4)\hat{i} - (0-10)\hat{j} + (6-5)\hat{k}$$

$\vec{v} = \langle 4, 10, 1 \rangle$

Great

$\vec{r} = \langle 0,0,0 \rangle + m\langle 4,10,1 \rangle$

W

$x = -3t$ $\quad x = 5s$
$y = t$ $\quad y = -2s$
$z = 2t$ $\quad z = 0$

$\boxed{4x + 10y + z = 0}$

5. Find the unit tangent vector to the curve r(t) = <3cos t, 3sin t, π-t> at time π.

$r'(t) = \langle -3\sin t, 3\cos t, -1 \rangle$

W

$r'(\pi) = \langle 0, -3, -1 \rangle$

$|r'(\pi)| = \sqrt{(0)^2 + (-3)^2 + (-1)^2} = \sqrt{10}$

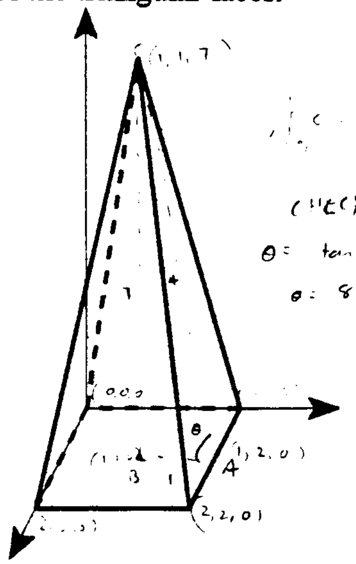
For unit vector divide everything by this number

unit tangent vector $= \langle 0, \frac{-3}{\sqrt{10}}, \frac{-1}{\sqrt{10}} \rangle$

Nice

W

6. As a monument to the annoying corporate spokesdog that died in problem 3, generous OU alumni plan to erect a monument (on the site of the last bit of open lawn on the OU campus, of course). For reasons known only to themselves, they want to build a seven meter tall pyramid topped by a chihuahua with glowing red demon-eyes. The pyramid will be shaped like the one below (shown without the chihuahua of course, for reasons of artistic secrecy) with vertices at the origin, (2,0,0), (0,2,0), (2,2,0), and its peak at (1,1,7). Find the angle formed between the base and one of the triangular faces.



check:

$\theta = \tan^{-1} 7$

$\theta = 81.9°$

$\theta = ?$ $AC = 0, 1, -7$

$AB \quad 0, 1, 0$

$\dfrac{\vec{AC} \cdot \vec{AB}}{|AC|\,|AB|}$

$\cos^{-1} \dfrac{(0 + 1 + 0)}{\sqrt{50} \cdot 1} = \theta$

$\boxed{\theta = 81.9°}$

Excellent

W

7. Prove that for every pair of vectors **a** and **b**, **b** is orthogonal to **a** × **b**.

$\vec{a} = \langle a_1, a_2, a_3 \rangle \quad \vec{b} = \langle b_1, b_2, b_3 \rangle$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \hat{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle$$

$$b \cdot (a \times b) = (b_1a_2b_3 - b_1a_3b_2) + (b_2a_3b_1 - b_2a_1b_3) + (b_3a_1b_2 - b_3a_2b_1) = 0$$

A dot product of two vectors that equals 0 indicates that these two vectors are orthogonal to one another.

Excellent

8. Reduce the equation $4x^2 - 9y^2 + 36y + z^2 = 0$ to one of the standard forms, classify the surface, and sketch its graph (be sure to label which axis is which!). Include on your graph the coordinates of at least two points on the surface.

10

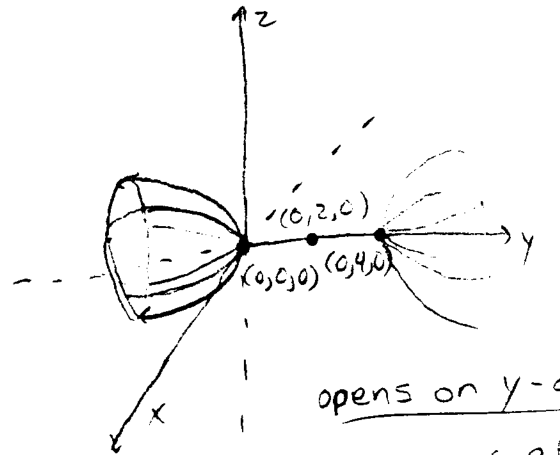
$4x^2 - 9y^2 + 36y + z^2 = 0$

$4x^2 - 9(y^2 - 4y + 4) + z^2 = 0 - 36$

$4x^2 - 9(y-2)^2 + z^2 = -36$

$$\boxed{-\frac{x^2}{9} + \frac{(y-2)^2}{4} - \frac{z^2}{36} = 1}$$

$\boxed{\text{hyperboloid of 2 sheet}}$



opens on y-axis

1 sheet opens at (0,0,0) in -y dir.

2 sheet opens at (0,4,0) in +y dir

Excellent

9. Ken says "Dude, on this Calc test there was this problem about, like, one of those para-things, like, you know, like it's the shape of a crystal or something? And there's this formula for the volume, like with vectors and cross and dot stuff. But I think I totally blew it because I kept getting zero, and you know there's no way, 'cause the vectors gotta make some volume, right?"

10

Either explain in clearer terms than Ken (doesn't take much!) why the formula for the volume of a parallelepiped must result in a value other than zero, or explain (clearly enough for Ken to understand) how such a thing might happen.

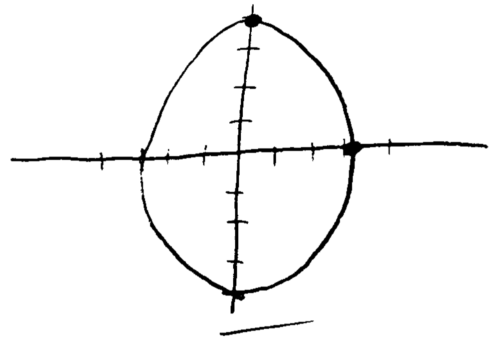
Such a thing might happen if All 3 vectors are in the same plane. the formula is $\langle A \times B \rangle \cdot C$

$A \times B$ gives a vector $\perp$ to A and B if C is in the same plane then $\langle A \times B \rangle \cdot C = 0$

Excellent and right on.

10. The ellipse $\frac{x^2}{9}+\frac{y^2}{16}=1$ (if we think of it as existing in the xy-plane) can be parametrized by $x=3\cos t$, $y=4\sin t$, $z=0$. Find the curvature of this ellipse at the points (3,0) and (0,4).

10



$r(t)=\langle 3\cos t, 4\sin t, 0\rangle$

$r'(t)=\langle -3\sin t, 4\cos t, 0\rangle$

$r''(t)=\langle -3\cos t, -4\sin t, 0\rangle$

$$\frac{|r'(t)\times r''(t)|}{|r'(t)|^3}$$

$$\frac{|\langle 0-0, 0-0, 12\sin^2 t - -12\cos^2 t\rangle|}{\left(\sqrt{(-3\sin t)^2+(4\cos t)^2+0^2}\right)^3} \Rightarrow \frac{|\langle 0,0,12(\sin^2 t+\cos^2 t)\rangle|}{|\langle 0,0,12\rangle|}$$

$$\frac{12}{\left(\sqrt{9\sin^2 t+16\cos^2 t}\right)^3}$$

$t=\cos^{-1}\frac{x}{3}$ @(3,0), $t=0$

$t=\sin^{-1}\frac{y}{4}$ @(0,4), $t=\frac{\pi}{2}$

For (3,0): $\frac{12}{64}=\frac{3}{16}$

For (0,4): $\frac{12}{27}=\frac{4}{9}$

Very nicely done!

Extra Credit (5 points possible):

The paraboloid $z=x^2+y^2$ is sliced by the plane $z=2x+2y$. What shape is the intersection?

+3

$2x+2y=x^2+y^2$

$0=x^2-2x+\_\_ +y^2-2y+\_\_$

$=x^2-2x-1 \quad +y^2-2y+1$

$2=(x-1)^2+(y-1)^2$

circle w/ radius $\sqrt{2}$ and [illegible])

Yes, sort of. That's the top view, the trick is it's slanted.