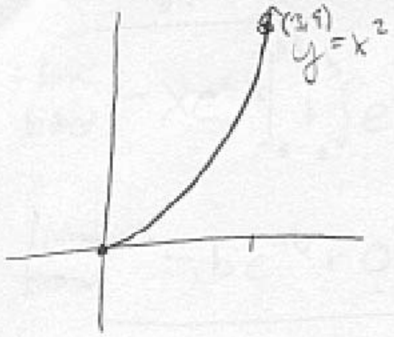
Exam 2b **Calc 2** **3/21/2003**

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. **Write an integral** for the length of the curve $y = x^2$ between (0,0) and (3,9).



$y' = 2x$

$$AL = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$\boxed{AL = \int_0^3 \sqrt{1 + (2x)^2}\; dx}$$

Nice

2. **Integrate** $\int x \cot^2\left(x^2\right) dx$ [Hint: Our table of integrals says $\int \cot^2 u\, du = -\cot u - u + C$].

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$\int x \cot^2(x^2) dx$ $\qquad u = x^2$

$\qquad du = 2x\, dx$

$\int x \cot^2(u) \frac{du}{2x}$ $\qquad dx = \frac{du}{2x}$

$\frac{1}{2} \int \cot^2 u\, du$

Great

$\frac{1}{2}[-\cot u - u + C] = \boxed{\frac{1}{2}\left[-\cot(x^2) - x^2 + C\right]}$

– 3. Show that $\int_0^\infty x\cdot e^{-x}\,dx = 1$.

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$$\int_0^\infty xe^{-x}\,dx = 1$$

$u = x \qquad v = -e^{-x}$

$du = dx \qquad dv = e^{-x}\,dx$

$$\lim_{b\to\infty}\int_0^b xe^{-x}\,dx = 1$$

$$\lim_{b\to\infty}\left[-xe^{-x} + \int e^{-x}\,dx\right]_0^b$$

$$\lim_{b\to\infty}\left[\frac{-x}{e^x} + \left(-e^{-x}\right)\right]_0^b$$

$$\lim_{b\to\infty}\left[\frac{-x}{e^x} + \frac{-1}{e^x}\right]_0^b$$

$$\lim_{b\to\infty}\left[\left(\frac{-b}{e^b} - \frac{1}{e^b}\right) - (0-1)\right]$$

$$\lim_{b\to\infty}\left[\frac{-b}{e^b} - \frac{1}{e^b} + 1\right] \overset{?}{=} 0 + 0 + 1 = \boxed{1}$$

$\lim_{b\to\infty} \overset{L'H}{=} \frac{-1}{e^b} = 0$ Yes!

$= 0$

4. Line 65 in our Table of Integrals says $\int \tan^2 u\,du = \tan u - u + C$. Suppose someone tells you that they suspect there's a typographical error there, because integrals should raise the degree of things rather than decrease it. **Verify** that the table entry is correct or erroneous.

the derivative of $\tan u - u = \sec^2 u - 1$

using the trig. identity $\tan^2 x + 1 = \sec^2 x$, the equation $\sec^2 u - 1$ can be rearranged as $\tan^2 u$. This table of the integral is correct since the derivative of $\tan u - u + c$ is equal to the equation needing integration.

Excellent!

5. **Integrate** $\int \frac{1}{9-x^2}\,dx$ [Hint: The answer can be written $\frac{-1}{6}\ln\left|\frac{x-3}{x+3}\right| + C$].

$-\int \frac{1}{x^2-9}\,dx = -\int \frac{1}{(x-3)(x+3)}\,dx$

$(x-3)(x+3)\left(\frac{1}{(x-3)(x+3)}\right) = \left(\frac{A}{(x-3)} + \frac{B}{(x+3)}\right)(x-3)(x+3)$

$1 = A(x+3) + B(x-3)$

If $x=3$ $\quad 1 = 6A + B(0)$

$A = 1/6$

If $x=-3$ $\quad 1 = A(0) + B(-6)$

$B = -1/6$

$= -\int \frac{1}{6(x-3)} - \frac{1}{6(x+3)}\,dx$

$= -1/6\left[\ln|x-3| - \ln|x+3|\right] + c$

$= -1/6 \ln\left|\frac{x-3}{x+3}\right| + c$

10

Well done

6. Derive the formula $\int \sqrt{2au - u^2}\, du = \frac{u-a}{2}\sqrt{2au-u^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{u-a}{a}\right) + C$ [You're welcome to use the integral formula $\int \sqrt{a^2 - u^2}\, du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$ if you find it convenient].

$= \int \sqrt{(-1)(u^2 - 2au)}\, du$

$= \int \sqrt{(-1)(u^2 - 2au + a^2 - a^2)}\, du$

$= \int \sqrt{(-1)[(u-a)^2 - a^2]}\, du$

$= \int \sqrt{a^2 - (u-a)^2}\, du$

$u + v = u - a$

$dv = du$

$= \int \sqrt{a^2 - v^2}\, dv$

$= \frac{v}{2}\sqrt{a^2 - v^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{v}{a}\right) + C$

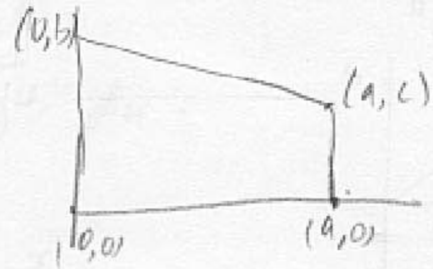
$= \frac{u-a}{2}\sqrt{2au - u^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{u-a}{a}\right) + C$

Well done!

7. **Find** the x coordinate of the centroid of the trapezoidal region with vertices at (0,0), (a,0), (0,b), and (a,c).

$$\bar{x} = \frac{\int x f(x)\,dx}{\int f(x)\,dx}$$

$$\bar{y} = \frac{\int \frac{1}{2}(f(x))^2\,dx}{\int f(x)\,dx}$$



$$\frac{y-b}{x} = \frac{c-b}{a}$$

$$y = \left(\frac{c-b}{a}\right)x + b$$

$$\bar{x} = \frac{\int_0^a x\left[\left(\frac{c-b}{a}\right)x+b\right]dx}{\int_0^a \left(\left(\frac{c-b}{a}\right)x+b\right)dx}$$

Yes

$$= \frac{\frac{c-b}{a}\int_0^a x^2\,dx + b\int_0^a x\,dx}{\frac{c-b}{a}\int_0^a x\,dx + \int_0^a b\,dx}$$

$$= \frac{\left(\frac{c-b}{a}\right)\cdot\frac{a^3}{3} + b\cdot\frac{a^2}{2}}{\frac{c-b}{a}\cdot\frac{a^2}{2} + b\cdot a}$$

$$= \frac{a^2}{a}\,\frac{\left[\frac{c-b}{3}+\frac{b}{2}\right]}{\frac{c-b}{2}+b}$$

$$= a\left[\frac{2c-2b+3b}{6} \times \frac{2}{c-b+2b}\right]$$

Excellent

$$= \frac{a}{3}\left[\frac{2c+b}{c+b}\right]$$

8. **Find** the surface area of the cone obtained by rotating the segment of the line $y=\frac{r}{h}x$ on [0,h] around the x-axis.

$$S.A = \int_0^h 2\pi \cdot f(x)\sqrt{1+[f'(x)]^2}\,dx$$

$$f(x) = \frac{r}{h}x$$

$$f'(x) = \frac{r}{h}$$

$$[f'(x)]^2 = \frac{r^2}{h^2}$$

$$S = \int_0^h 2\pi \cdot \frac{r}{h}x \cdot \sqrt{1+\frac{r^2}{h^2}}\,dx = 2\pi\frac{r}{h}\int_0^h x\sqrt{\frac{h^2+r^2}{h^2}}\,dx$$

$$= 2\pi\frac{r}{h}\cdot\frac{\sqrt{h^2+r^2}}{h}\int_0^h x\,dx$$

$$= \frac{\cancel{2}\pi r\sqrt{h^2+r^2}}{h^2}\cdot\frac{x^2}{\cancel{2}}\Big]_0^h = \frac{2\pi r\sqrt{h^2+r^2}}{\cancel{h^2}}\cdot h^2$$

$$= \pi r\sqrt{h^2+r^2}$$

Surface area $= \pi r\sqrt{h^2+r^2}$

Great Job

10. Given that $\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$, **find** $\Gamma(½)$. [Remember that $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$].

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty t^{-1/2}e^{-t}\,dt$$

Let $t = u^2$

$dt = 2u\,du$

$$= \int_0^\infty \cancel{u^{-1}}\cdot e^{-u^2}\cdot 2\cancel{u}\,du$$

$$= 2\int_0^\infty e^{-u^2}\,du = 2\times\frac{1}{2}\sqrt{\pi} = \boxed{\sqrt{\pi}}$$

Nice.

9. Brandi is a calculus student at E.S.U. who's having some trouble with improper integrals. Brandi says "So there was this problem on our test, and it was to say if $\int_1^\infty \frac{1}{x \ln x}\,dx$ converged or not. I drew this total blank on how to find the antiderivative, but I thought about it and decided I didn't really have to. See, you know that if you integrate 1/x it's infinity, but if you integrate $1/x^2$ or anything else where the denominator is more than just x it converges. So I just said all that and said it must converge. But the grader gave me no points at all, and just said that didn't work, but not why, and then he told me to go away because he hates dealing with students."

Explain to Brandi either why she's right, or what's wrong with her reasoning.

So Brandi, you had a good idea, that if you can't work out the integral you could just compare it to some other integral that you _do_ know. The trouble in this case is that comparing it to $\int_1^\infty \frac{1}{x}dx$ just tells you it's less than something infinite, which isn't much help. The same trouble happens if you compare it to $\int_1^\infty \frac{1}{x^2}dx$, except then you find out it's bigger than something finite, which still leaves too many possibilities. So those weren't wrong things to try, but they turn out not to actually tell you anything.

For big enough x:

$$x \cdot \ln x > x$$

$$\Rightarrow \frac{1}{x \cdot \ln x} < \frac{1}{x}$$

$$\Rightarrow \int_{10}^\infty \frac{1}{x \cdot \ln x}\,dx < \int_{10}^\infty \frac{1}{x}\,dx$$

But besides all that, Brandi, you made the worst mistake possible with improper integrals. That integrand isn't continuous when $x = 1$, so you really have to break it into two parts and deal with limits as $a \to 1^+$ and $b \to \infty$. What you did totally ignored the discontinuity at the left end of the interval, and that's probably why they graded you so harshly.