Exam 4 Calculus 2 5/9/2003

Each problem is worth 10 points, show all work and give adequate explanations for full credit. Please keep your work as legible as possible.

1. (a) **Write the first four terms** in the sequence $a_n = 1/(2n+1)$.

(b) **Write the first four partial sums** of the series $\sum_{n=1}^{\infty} \frac{1}{2n+1}$.

a) $a_1 = \frac{1}{2+1} = \frac{1}{3}$ $\quad a_3 = \frac{1}{2\cdot 3+1} = \frac{1}{7}$

$a_2 = \frac{1}{2\cdot 2+1} = \frac{1}{5}$ $\quad a_4 = \frac{1}{2\cdot 4+1} = \frac{1}{9}$ Sequence $= \{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}\}$

10 b) $S_1 = \frac{1}{2\cdot 1+1} = \frac{1}{3}$

$S_2 = \frac{1}{3} + \frac{1}{2\cdot 2+1} = \frac{1}{3} + \frac{1}{5}$ Good

$S_3 = \frac{1}{3} + \frac{1}{5} + \frac{1}{2\cdot 3+1} = \frac{1}{3} + \frac{1}{5} + \frac{1}{7}$

$S_4 = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{2\cdot 4+1} = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9}$ $\quad$ ~~Series~~ $= \frac{1}{3} +$

2. **Write the Taylor polynomial** of degree 6 for $f(x) = \sin x$ centered at $a = \pi/2$.

$f(x) = \sin x$ $\quad f(\frac{\pi}{2}) = \sin\frac{\pi}{2} = 1$

$f'(x) = \cos x$ $\quad f'(\frac{\pi}{2}) = \cos\frac{\pi}{2} = 0$

$f''(x) = -\sin x$ $\quad f''(\frac{\pi}{2}) = -\sin\frac{\pi}{2} = -1$

10 $f^{(3)}(x) = -\cos x$ $\quad f^{(3)}(\frac{\pi}{2}) = -\cos\frac{\pi}{2} = 0$

$f^{(4)}(x) = \sin x$ $\quad f^{(4)}(\frac{\pi}{2}) = \sin\frac{\pi}{2} = 1$

$f^{(5)}(x) = \cos x$ $\quad f^{(5)}(\frac{\pi}{2}) = \cos\frac{\pi}{2} = 0$

$f^{(6)}(x) = -\sin x$ $\quad f^{(6)}(\frac{\pi}{2}) = -\sin\frac{\pi}{2} = -1$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x-a)^n}{n!}$$

$$= 1 + (0)(x-\tfrac{\pi}{2}) + \frac{(-1)(x-\frac{\pi}{2})^2}{2!} + \frac{(0)(x-\frac{\pi}{2})^3}{3!} + \frac{(1)(x-\frac{\pi}{2})^4}{4!} + \frac{(0)(x-\frac{\pi}{2})^5}{5!} + \frac{(-1)(x-\frac{\pi}{2})^6}{6!}$$

$$f^6(x) = 1 - \frac{(x-\frac{\pi}{2})^2}{2!} + \frac{(x-\frac{\pi}{2})^4}{4!} - \frac{(x-\frac{\pi}{2})^6}{6!}$$

Great

3. **Give an example** of a series which is convergent but not absolutely convergent.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent because:

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent as it is an alternating harmonic series

but $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right|$ is divergent as it forms a harmonic series.

Great

4. **Give a power series expansion** for $\frac{1}{1-x^5}$ in sigma notation.

$$\frac{1}{1-x} = \sum_{n=0} x^n$$

$$\therefore \frac{1}{1-x^5} = \sum_{n=0}^{\infty} (x^5)^n = \sum_{n=0}^{\infty} x^{5n}$$

$$= 1 + x^5 + x^{10} + x^{15} + \cdots$$

Nice Work

5. **Show** that the radius of convergence of the series $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$ is infinite.

Use ratio test

will fall out due to absolute value signs

$$\lim_{n\to\infty} \left| \frac{x^{2(n+1)}}{(2(n+1))!} \cdot \frac{(2n)!}{x^{2n}} \right|$$

$$\lim_{n\to\infty} \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}} \right|$$

$$\lim_{n\to\infty} |x^2| \left(\frac{1}{(2n+2)(2n+1)} \right)$$

$$\lim_{n\to\infty} |x^2| \left(\frac{\frac{1}{n}}{\underbrace{(2+\frac{2}{n})(2+\frac{1}{n})}_{\text{go to 0 as } n \text{ approaches } \infty}} \right) \leftarrow \boxed{\text{goes to 0 as } n \text{ approaches } \infty}$$

Excellent

$\lim_{n\to\infty} 0$ for any value of x, so the radius of convergence of the series is infinite.

6. **Show** whether the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}}$ converges or diverges.

$(n^2+n)(n+2)$

$n^3+2n^2+n^2+2n$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+3n^2+2n}}$$

$$\sqrt{n^3+3n^2+2n} > \sqrt{n^3}$$

$$\frac{1}{\sqrt{n^3+3n^2+2n}} < \frac{1}{\sqrt{n^3}} \leftarrow$$ power series with $p = 3/2 > 1$, so it will converge

Wonderful!

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}} < \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$ which is a convergent power series, then the given series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)(n+2)}}$ must also converge according to the comparison test since its series is less than a known convergent series.

7. Find the **interval of convergence** of the series $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

Rat. Test: $\lim_{n\to\infty} \left| \frac{\frac{(-3)^{n+1} x^{n+1}}{\sqrt{(n+1)+1}}}{\frac{(-3)^n x^n}{\sqrt{n+1}}} \right| = \lim_{n\to\infty} \left| \frac{\frac{3^{n+1} x^{n+1}}{\sqrt{n+2}}}{\frac{3^n x^n}{\sqrt{n+1}}} \right|$

$= \lim_{n\to\infty} \left| \frac{3^{n+1}}{3^n} \cdot \frac{x^{n+1}}{x^n} \cdot \frac{\sqrt{n+1}}{\sqrt{n+2}} \right|$

$= |3x|$ Exactly!

Converges when $|3x| < 1$ or $|x| < \frac{1}{3}$

$\sum \frac{(-3)^n (\frac{1}{3})^n}{\sqrt{n+1}} = \sum \frac{(-1)^n}{\sqrt{n+1}}$

$\frac{1}{\sqrt{n+2}} \overset{?}{<} \frac{1}{\sqrt{n+1}}$

$\sqrt{n+1} < \sqrt{n+2}$

$n+1 < n+2$

$1 < 2$

$\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$

Converges on alt. Series Test Good

$\sum \frac{(-3)^n (-\frac{1}{3})^n}{\sqrt{n+1}} = \sum \frac{1}{\sqrt{n+1}}$

$\sum \frac{1}{n}$ is a series that diverges

$\frac{1}{n} \leq \frac{1}{\sqrt{n+1}}$

$\sqrt{n+1} \leq n$

$n+1 \leq n^2$

$1 \leq n^2 + n$

Nice

Diverges based on comparison Test.

I. o. C. is when $-\frac{1}{3} < x \leq \frac{1}{3}$

Very nicely done!

8. For what values of p does the series $\sum_{n=1}^{\infty} n\left(1+n^2\right)^p$ converge?

Int. Test: $\int_1^{\infty} x(1+x^2)^p\,dx$ let $u = 1+x^2$

$\frac{du}{dx} = 2x$

$= \frac{1}{2}\int_2^{\infty} u^p\,du$

$\frac{du}{2x} = dx$

We know from before this integral converges for $p < -1$ and diverges for $p \geq -1$, so by the int. test the same is true of this series.

9. Chaz is a calculus student at E.S.U. who's having some trouble with series. Chaz says "Dude, this calc stuff is totally destroying me. Like, I figured those series would pretty much be just common sense, but I totally bombed the test. But there's this one where I think I got screwed, because my way totally makes sense. I showed how this one series they asked about, it was like one over n minus one over the whole n minus 1, so it was a thing that diverged minus a thing that diverged, and that meant it had to diverge totally, right? But the grader gave me, like, no credit, and wrote this stuff about how that's bad math and stuff. I think he just hates me because I'm in a frat."

Only with finite #'s not infinite numbers.

Explain clearly, in terms Chaz can understand, either why his approach is incorrect or why it's valid and how he can convince the grader.

$$\sum\Big(\underbrace{\frac{1}{n}}_{\text{Diverges}} - \underbrace{\frac{1}{n-1}}_{\text{Diverges}}\Big) = ?$$

Diverges

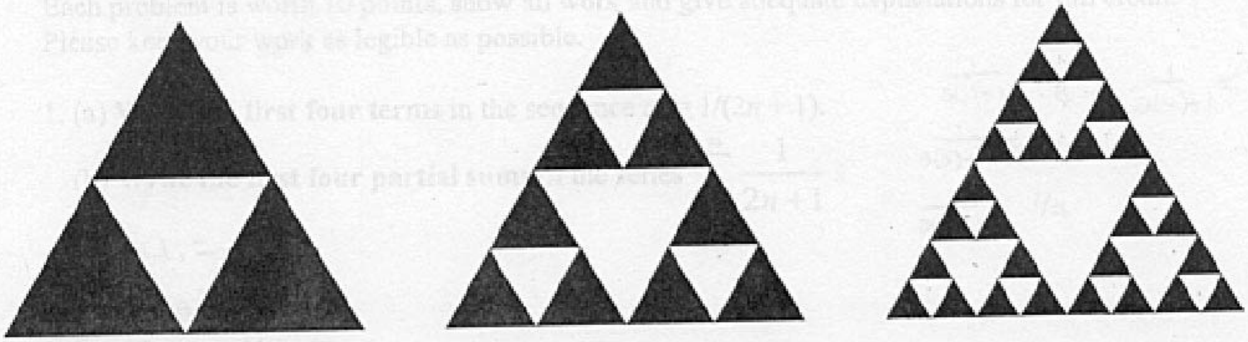
Chaz is right in saying $\sum \frac{1}{n}$ & $\sum \frac{1}{n-1}$ diverge, but you can't say $\infty - \infty = \infty$, the infinity signs don't work like regular numbers.

Exactly.

He can find the correct answer by making a common fraction out of $\frac{1}{n} - \frac{1}{n-1}$ of finding the infinite sum of that.

$\sum\left(\frac{1}{n} - \frac{1}{n-1}\right) = \sum\left(\frac{-1}{n(n+1)}\right)$ is what Chaz wants the answer of.

10. Begin with an equilateral triangle with a total area of 1 and successively remove smaller triangles from it in the manner shown below:



(a) If a_n is the total area removed in step n alone, find a_1, a_2, a_3, and a_4.

$a_1 =$ 1/4

$a_2 =$ 3/16

$a_3 =$ 9/64

$a_4 =$ 27/256

(b) If we continue the process indefinitely, express the total area removed as a series and find the sum of that series.

well, this is a geom series w/ $a = 1/4$

and $r = 3/4$

so the series is: $\boxed{\frac{1}{4}\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^{n-1}}$

Excellent!

the series is conv. because $|3/4| < 1$ so the sum is:

$$\frac{a}{1-r} = \frac{1/4}{1-3/4} = \frac{1/4}{1/4} = 1$$