Exam 1 **Differential Equations** **3/7/2003**

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. **Show** that $y(t) = e^{-t} + t^2$ is a solution to the differential equation $\dfrac{dy}{dt} + y = t^2 + 2t$.

$y(t) = e^{-t} + t^2$

$\frac{dy}{dt} = -e^{-t} + 2t$

so,

$(-e^{-t} + 2t) + (e^{-t} + t^2)$

$= t^2 + 2t$

$t^2 + 2t = t^2 + 2t$

Both sides are equal so the equation is a solution.

Exactly

2. Given that $S(t) = \dfrac{30t + t^2 + C}{15 + t}$ is a general solution to a differential equation, **find a particular solution** which satisfies the initial condition $S(0) = 3$.

we set $S(0) = 3$

$\Rightarrow 3 = \dfrac{30 \cdot 0 + 0^2 + C}{15 + 0}$

$\Rightarrow C = 45$

Great

$\therefore$ Particular soln. would be $S(t) = \dfrac{30t + t^2 + 45}{15 + t}$ $\boxed{\text{Ans}}$.

3. **Give an example** of a differential equation which is not linear.

$$\frac{dy}{dt} = y^2$$

For a differential equation to be linear, the y term (in my case y) must be to the power of one. Since it is squared in my equation, it is not linear.

Exactly.

4. An enormous vat begins with 50 gallons of pure water at time $t = 0$. More pure water is added from one spigot at the rate of 5 gallons per minute. Meanwhile, a second spigot adds salt water with a concentration of 0.5 pounds of salt per gallon at a rate of 4 gallons per minute. The solution is kept well mixed, and 3 gallons per minute are drained out from the bottom of the tank. **Write a differential equation** for the rate of change of the amount of salt in the tank over time.

$$\frac{.5 \text{ lb salt}}{\text{gal}} \cdot \frac{4 \text{ gal}}{\text{min}} = \frac{2 \text{ lb}}{\text{min}}$$

$$\frac{dW}{dt} = 5 + 4 - 3 = 6 \frac{\text{gal}}{\text{min}}$$

amount water (t)

$$50 + 6t$$

$$\frac{dS}{dt} = 2 - \left(\frac{S}{50+6t}\right)(3)$$

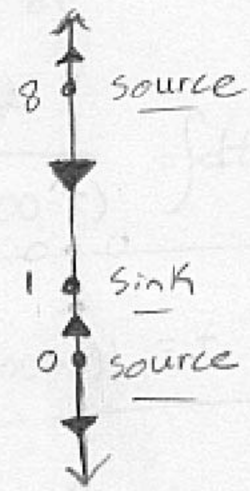
$$\frac{\text{lb}}{\text{min}} - \frac{\text{lb}}{\cancel{\text{gal}}} \cdot \frac{\cancel{\text{gal}}}{\text{min}}$$

Great

$$\boxed{\frac{dS}{dt} = 2 - \frac{3S}{50+6t}}$$

5. **Sketch the phase line** for the differential equation $\dfrac{dy}{dt} = 0.05t(t-1)(t-8)$ and label each equilibrium point as a sink, source, or node.

$\dfrac{dy}{dt} = .05t(t-1)(t-8)$



Excellent

$.05(.5)(-.5)(-7.5) = .0937$

$.05(2)(1)(-6) = -.6$

$.05(-1)(-2)(-9) = -.9$

$.05(10)(9)(2) = 9$

6. Consider the differential equation $\dfrac{dP}{dt} = 2P - \dfrac{P^2}{50} - 30$ for the population of fish in a lake if 10 fishing licenses are granted, with initial condition $P_0 = 100$. **Use Euler's Method** (with a step size of $\Delta t = 1$) to approximate the population at $t = 1$ and $t = 2$.

$t_{k+1} = t_k + \Delta t$ $\qquad$ $y_{k+1} = y_k + f(t_k, y_k)\Delta t$

$P_1 = P_0 + \left(2P_0 - \frac{P_0^2}{50} - 30\right)\Delta t$ $\qquad$ $\Delta t = 1$

Nice Job

$P_1 = 100 + \left(2(100) - \frac{100^2}{50} - 30\right)1$

$P_1 = 100 + (200 - 200 - 30)(1)$

$\boxed{P_1 = 70 \text{ at } t = 1 \qquad P_2 = 82 \text{ at } t = 2}$

$P_2 = 70 + \left(2(70) - \frac{70^2}{50} - 30\right)1$

$P_2 = 70 + (140 - 98 - 30)1$

$70 + 12$

7. A yam is placed in a hot oven at time $t = 0$ and begins to $\boxed{\text{warm up}}$ according to the differential equation $\dfrac{dT}{dt} = k(400 - T)$. **Find a general solution** $T(t)$ for the temperature of the yam after t minutes in the oven.

$$\frac{d\left[400 - Ce^{-kt}\right]}{dt} = kCe^{-kt}$$

$$k\left(400 - \left[400 - Ce^{-kt}\right]\right)$$

$$kCe^{-kt}$$

Great! equal, solution ✓s

$$\frac{dT}{dt} = k400 - kT$$

$$\frac{1}{k400 - kT}\frac{dT}{dt} = 1$$

$$-\frac{1}{k}\int \frac{-1}{400 - T}\frac{dT}{dt}\,dt = \int dt$$

$$-\tfrac{1}{k} \cdot \ln|400 - T| = t + C$$

constant · constant = constant

$$\ln|400 - T| = -kt + C$$

$\rightarrow$ e^{constant} = constant

C

$$|400 - T| = Ce^{-kt}$$

Well done

Since yam is warming up, $T < 400$, so $400 - T > 0$.

Yes!

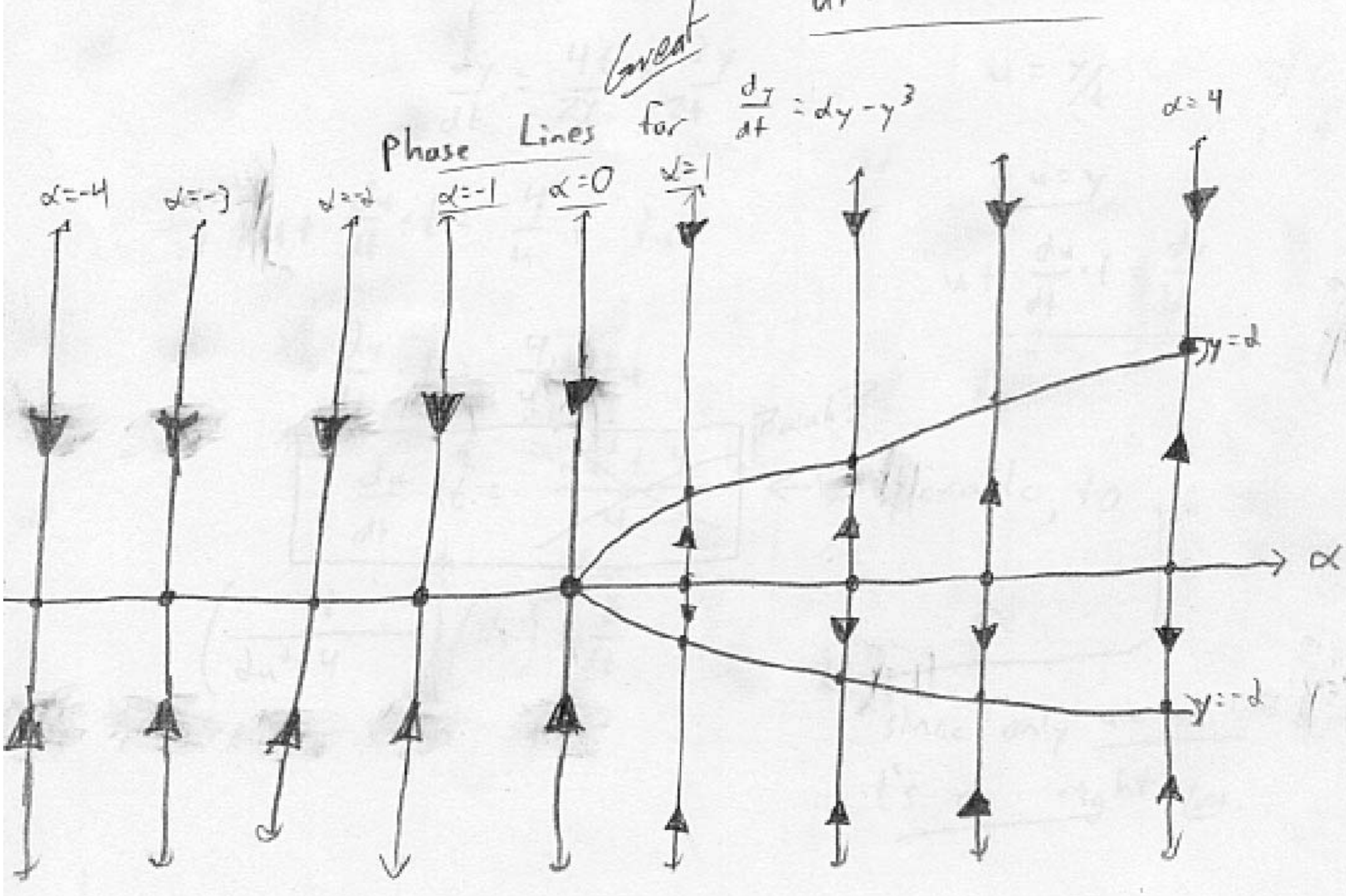
$$400 - T = Ce^{-kt}$$

$$T = 400 - Ce^{-kt}$$

$$\boxed{T(t) = 400 - Ce^{-kt}}$$

8. **Sketch the bifurcation diagram** for the differential equation $\frac{dy}{dt} = ay - y^3$, where a is the parameter.

$$\frac{dy}{dt} = y(\alpha - y^2)$$



9. The differential equation $2ty\dfrac{dy}{dt} = 4t^2 + 3y^2$ is not separable. Use the substitution $u = \dfrac{y}{t}$ to transform it into a new differential equation which is separable, and make it clear how you know it is separable. **You do not need to solve the new equation.**

$u = y/t \quad y = ut$

$\dfrac{dy}{dt} = \dfrac{du}{dt}t + u$

$$2t^2u\left(\frac{du}{dt}t + u\right) = 4t^2 + 3(ut)^2$$

$$2t^3u\frac{du}{dt} + 2t^2u^2 = 4t^2 + 3u^2t^2$$

$$2t^3u\frac{du}{dt} = 4t^2 + u^2t^2$$

$$\frac{du}{dt} = \frac{4t^2 + u^2t^2}{2t^3u}$$

Well done

$$\frac{du}{dt} = 2(tu)^{-1} + \tfrac{1}{2}ut^{-1}$$

$$\frac{du}{dt} = \frac{2u^{-1} + \frac{1}{2}u}{t}$$

$$(2u^{-1} + \tfrac{1}{2}u)^{-1}\,du = t^{-1}\,dt$$

this is it seperated

10. The amount of salt in a polluted pond is modeled by the differential equation

$\dfrac{dS}{dt} = \dfrac{5}{2} - \dfrac{26S}{200-t}$. **Find a general solution** to this differential equation.

$$\frac{dS}{dt} + \frac{26S}{200-t} = \frac{5}{2}$$

✓ Linear

$$u = e^{\int \frac{26S}{200-t}\, dt}$$

$$u = e^{-26 \ln|200-t|}$$

$$u = e^{\ln(200-t)^{-26}}$$

$$u = (200-t)^{-26}$$

W

$$u\frac{dS}{dt} + u\frac{26S}{200-t} = u\frac{5}{2}$$

$$(200-t)^{-26}\frac{dS}{dt} + (200-t)^{-27}\,26S = \frac{5}{2}(200-t)^{-26}$$

$$\int \frac{dS}{dt}\left[(200-t)^{-26} \cdot S\right] = \int \frac{5}{2}(200-t)^{-26}\, dt$$

$$\int u^{-26} = \frac{-1}{25} u^{-25}\, du$$

$$(200-t)^{-26} S = \frac{-5}{2}\left(\frac{1}{-25}\right)(200-t)^{-25} + C$$

$$(200-t)^{-26} S = \frac{1}{10(200-t)^{25}} + C$$

$$(200-t)^{-26}$$

Excellent!

$$\boxed{S = \frac{1}{10}(200-t) + C(200-t)^{26}}$$

Check

$$\frac{dS}{dt} = 0 + \frac{1}{10} - C26(200-t)^{25}$$

$$= \frac{5}{2} - \frac{26\left(\frac{1}{10}(200-t) + C(200-t)^{26}\right)}{200-t}$$

$$\checkmark = \frac{5}{2} - \frac{26}{10} - 26C(200-t)^{25}$$