Exam 1b Calc 2 2/22/2004

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Evaluate $\int_1^3 \frac{1}{x^2}\,dx$.

$x^n = \frac{x^{n+1}}{n+1}$

$\frac{1}{x^2} = x^{-2}$

$\int_1^3 x^{-2} = -\frac{x^{-1}}{1}\Big|_1^3$

$-\frac{1}{x}\Big|_1^3$

$\left(-\frac{1}{3}\right) - \left(-\frac{1}{1}\right)$

$-\frac{1}{3} + 1 = \boxed{\frac{2}{3}}$

Good

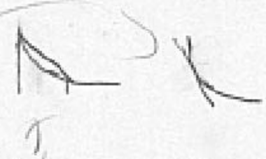
2. If the integral of a decreasing concave up function is approximated with the left, right, trapezoid and midpoint rules (with the same number of subdivisions) and the results (to three decimal places) are 0.601, 0.632, 0.633, and 0.664,

(a) Which rule matches with which approximation?

left is over
right is under

left = .664
right = .601
trap = .633
midpoint = .632

Good ✓

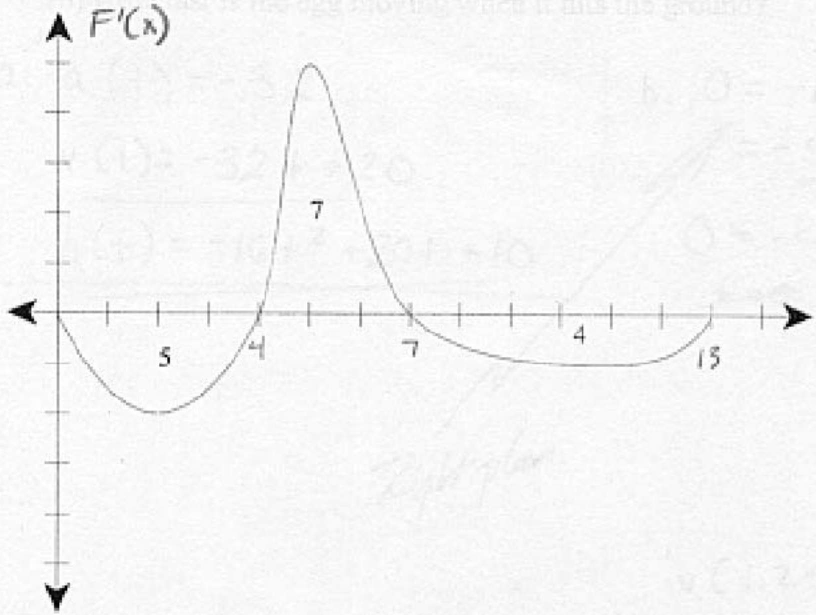


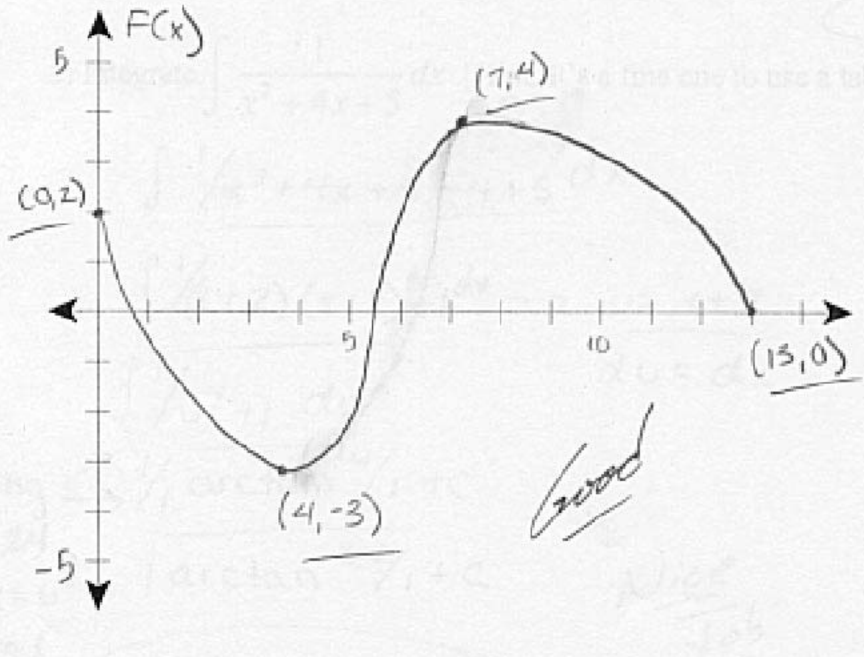
trap.s over approximation
middle is under approximation

(b) What would Simpson's approximation of the integral be?

$S = \frac{2M + T}{3} = \frac{2(.632) + .633}{3} = \boxed{.632333\ldots}$

3. Given the graph of $F'(x)$ shown below (with the areas of several regions marked) and the fact that $F(0) = 2$, sketch the graph of $F(x)$ and label the coordinates of all critical points on the graph of $F(x)$.





4. If an egg is thrown upward at 20 feet per second from a height of 10 feet [and acceleration due to gravity is 32 feet per second^2 downward],

(a) Find formulas for the egg's velocity and height after t seconds

(b) How fast is the egg moving when it hits the ground?

20 ft/s, 10

a) $h(x) = -16t^2 + V_0 t + H_0$

$v(x) = -32 \text{ ft/s } t + V_0$

$a(x) = -32 \text{ ft/s}^2$

$$\boxed{\begin{array}{l}\text{For velocity: } v(t) = -32 \tfrac{ft}{s} t + 20 \tfrac{ft}{s} \\ \text{For height: } h(t) = -16t^2 + 20t + 10\end{array}}$$

b) how fast $0 = -16t^2 + 20t + 10$

$$\frac{-20 \pm \sqrt{20^2 - 4 \cdot -16 \cdot 10}}{2 \cdot -16} = \frac{20 \pm \sqrt{1040}}{32} = 1.63 \text{ or } -0.38$$

Well done

$$t = \frac{20 + \sqrt{1040}}{32} \approx 1.633 \text{ sec}$$

$$v(t) = -32 \cdot (1.633) + 20 = \boxed{32.266 \text{ ft/sec downwards}}$$

5. Integrate $\displaystyle\int \frac{1}{x^2 + 4x + 5}\, dx$ [Hint: It's a fine one to use a table on with some adjustment].

$$= \int \frac{1}{(x^2 + 4x + 4) - 4 + 5}$$

$$= \int \frac{1}{(x+2)^2 + 1}$$

$u = x + 2$

$\frac{du}{dx} = 1 \quad \therefore \quad du = dx$

use line #24

$$= \int \frac{1}{u^2 + 1} = \frac{1}{1} \arctan \frac{u}{1} + C$$

Nice Job

$$= \arctan \frac{x+2}{1}$$

W

$$\boxed{= \arctan(x+2) + C}$$

6. Compute $\displaystyle\int_1^{\pi^2} \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$ exactly.

$u = x^{\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$dx = du\,\sqrt{x}(2)$

$2\int_1^{\pi^2} \frac{\cos u}{\sqrt{x}} \cdot du\,\sqrt{x}$

$2\int \cos u \longrightarrow 2[-\sin u]$

$-2[\sin(\sqrt{x})]\Big|_1^{\pi^2}$

Excellent

$-2[0] + 2[-\sin\sqrt{1}]$

$= \boxed{2\sin\sqrt{1}}$

7. Biff is a calculus student at Factory State University, and he's having some trouble with integrals. Biff says "Dude, I've got this homework I've gotta do for math tomorrow, and I'm totally stumped. It's a bunch of those intergals with the limit things on them, you know? And I thought it would be okay because I could use the table, right? But my roommate said you couldn't use the table because it doesn't work for ones with the limit things. So is that true?"

Explain clearly to Biff why it either is or is not appropriate to use a table of integrals for a definite integral.

Why wouldn't it be appropriate to use a table to find a definite integral? Just because the table doesn't show examples of definite integrals, doesn't mean you can't use them to find a definite integral. For example:
If we have an indefinite integral that reads

$\int \frac{1}{x^2+9}\,dx$, the table states that the answer is $\frac{1}{3} \arctan \frac{x}{3} + C$.

Now let's say we have the definite integral

$\int_0^4 \frac{1}{x^2+9}\,dx$, we can still use the table to find $\frac{1}{3} \arctan \frac{x}{3} + C$ & solve for the definite integral by $\frac{1}{3} \arctan \frac{4}{3} - \left(\frac{1}{3} \arctan \frac{0}{3}\right) =$

$\frac{1}{3} \arctan \frac{4}{3} + C$

Exactly

It can be used for both!

8. Use partial fractions to decompose $\dfrac{1}{(x-a)(x-b)}$ into simpler fractions (in terms of the constants a and b).

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

$1 = A(x-b) + B(x-a)$ Let $x = b, a$ $B = \dfrac{1}{b-a}$

whe $x = b \Rightarrow 1 = B(b-a)$ $A = \dfrac{1}{a-b}$

when $x = a \Rightarrow 1 = A(a-b)$

$$\frac{1}{(x-a)(x-b)} = \frac{\frac{1}{a-b}}{(x-a)} + \frac{\frac{1}{b-a}}{(x-b)}$$

Excellent

9. Derive the formula $\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C$.

$\int \underbrace{e^x}_{v'} \underbrace{\sin x}_{u} \, dx$

Let $u = \sin x$ $\quad v = e^x$

$u' = \cos x$ $\quad v' = e^x$

we have,

$\int uv' = uv - \int u'v$

Therefore

$$\int e^x \sin x \, dx = \sin x \cdot e^x - \int \cos x \cdot e^x \, dx$$

$$= e^x \sin x - \int e^x \cos x \, dx$$

Again, using integration method for $\int \underbrace{e}_{v'}{}^x \underbrace{\cos x}_{u} \, dx$

$u = \cos x$ $\quad v = e^x$

$u' = -\sin x$ $\quad v' = e^x$

$$\int e^x \cos x \, dx = \cos x \cdot e^x - \int -\sin x \cdot e^x \cdot dx$$

$$= e^x \cos x + \int e^x \sin x \cdot dx$$

So,

$$\int e^x \sin x \, dx = e^x \sin x - \left[e^x \cos x + \int e^x \sin x \, dx \right]$$

$$\int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

$$\int e^x \sin x \, dx + \int e^x \sin x \, dx = e^x (\sin x - \cos x) + C$$

$$2 \int e^x \sin x \, dx = e^x (\sin x - \cos x) + C$$

$$\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C \quad \text{Ans}$$

Very nicely done!

10. The area of an ellipse is given by $\int_0^a 4b\sqrt{1-\frac{x^2}{a^2}}\,dx$, where the constants a and b represent the major and minor radii of the ellipse. Use the substitution $x = a \sin \theta$ to show that the value of this integral is πab. [Once you've carried out the substitution, feel free to use the table for the resulting trig integral].

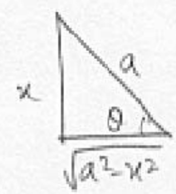
$$\int_0^a 4b\sqrt{1-\frac{x^2}{a^2}}\,dx$$

Let, $x = a\sin\theta$

$\frac{dx}{d\theta} = a\cos\theta$ or, $dx = a\cos\theta\, d\theta$

$\sin\theta = \frac{x}{a}$

$\theta = \sin^{-1}(x/a)$



$$= \int_0^a 4b\sqrt{1-\frac{a^2\sin^2\theta}{a^2}}\; a\cos\theta\, d\theta$$

$$= \int_0^a 4b\sqrt{1-\sin^2\theta}\; a\cos\theta\, d\theta$$

$\cos\theta = \frac{\sqrt{a^2-x^2}}{a}$

$$= 4ab\int_0^a \cos^2\theta\, d\theta$$

$$= 4ab\left[\frac{1}{2}\cos\theta\sin\theta + \frac{1}{2}\int d\theta\right]_0^a$$

Beautifully done

$$= 4ab\left[\frac{1}{2}\cos\theta\sin\theta + \frac{1}{2}\theta\right]_0^a$$

$$= 4ab\left[\frac{1}{2}\frac{\sqrt{a^2-x^2}}{a}\cdot\frac{x}{a} + \frac{1}{2}\sin^{-1}(x/a)\right]_0^a$$

$$= 4ab\left[\frac{1}{2}\frac{\sqrt{a^2-a^2}}{a}\cdot\frac{a}{a} + \frac{1}{2}\sin^{-1}(a/a) - \frac{1}{2}\frac{\sqrt{a^2-0}}{a}\cdot\frac{0}{a} - \frac{1}{2}\sin^{-1}(0/a)\right]$$

$$= 4ab\left[\frac{1}{2}\sin^{-1}(1) - \frac{1}{2}\sin^{-1}(0)\right]$$

$$= 4ab\cdot\frac{1}{2}\left(\frac{\pi}{2}-0\right) = \boxed{\pi ab}$$