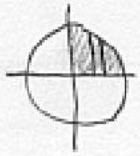
Exam 2 Calc 2 2/20/2004

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. **Set up** an integral for the area of the first-quadrant portion of a circle centered at the origin with a radius of 5.



Radius = 5

width = $\sqrt{25-x^2}$

area = $\sqrt{25-x^2}\,\Delta x$

$x^2+y^2=5^2$

$x^2+y^2=25$

$y=\sqrt{25-x^2}$

$x=\sqrt{25-y^2}$

total area

$$\int_0^5 \sqrt{25-x^2}\,dx$$

Great

2. Jon is considering a big investment in black-market software distribution. He thinks the investment will pay \$5000 per year initially, but drop off linearly to \$0 after 2 years. **Set up an integral** that gives a fair price to pay for this investment today (as an alternative to investing the money at 6% continuously compounded interest)?

x	y
0	5000
2	0

$m = \frac{5000-0}{0-2} = -2500$

$y = mx + b$

$P(t) = -2500x + 5000$

$$\boxed{\int_0^2 (-2500x + 5000)\, e^{-.06t}\, dt}$$

Excellent

3. A spring with natural length $\boxed{40\text{cm}}$ requires 5N of force to hold it stretched to a length of 50cm. **How much work** is done in stretching the spring from natural length to a length of $\boxed{60\text{cm}}$?

$$5N = K \cdot (0.5 - 0.4)m \Rightarrow K = 50 \ N/m$$

$$\therefore \text{work done} = \int_0^{0.2} 50x \, dx$$

$$\Rightarrow 50 \cdot \frac{x^2}{2} \Big|_0^{0.2}$$

$$\Rightarrow 25 \cdot x^2 \Big|_0^{0.2}$$

$$\Rightarrow 1 \ J$$ Good

4. Suppose that the probability density function for the failure of the airbag system in Jon's car after t months is given by $p(t) = ce^{-ct}$ for values of t greater than zero (and by zero otherwise).

(a) If the probability of failure within the first month is 40%, **find the value** of c to two decimal places.

$$\int_0^1 ce^{-ct} = 0.4 \Rightarrow C \times \frac{1}{-C} \left[\frac{1}{e^c} - \frac{1}{e^0}\right] = 0.4$$

$$\text{or}, -\frac{1}{e^c} + \frac{1}{e^0} = 0.4 \text{ or } \frac{1}{e^c} = 0.6 \text{ or } e^{-c} = 0.6 \text{ or } -c \ln e = \ln 0.6$$

$$\therefore c = 0.51$$

(b) **Write an integral** which expresses the probability of the airbag system failing within the first 6 months.

$$p(t) = \int_0^6 0.51 \, e^{-0.51t} \, dt.$$ Excellent

5. A thin metal rod 5 inches long has density given by $\delta(x) = \sqrt{4+x}$ at a point x inches from its left end. **Set up an integral** and use it to find the mass of the rod.

D=2 $\sqrt{3}$

5

L=0

Mass = $\sqrt{4+x} \cdot \Delta x$

mass $\int_0^5 \sqrt{4+x}\, dx$

$\int_0^5 (4+x)^{1/2}\, dx$

$u = 4+x$

$\frac{du}{dx} = 1$

$\int_0^5 u^{1/2}\, du$

$\frac{2}{3} u^{3/2} \Big|_0^5$

Excellent

$\frac{2}{3} (4+x)^{3/2} \Big|_0^5$

$18 - 5.3\overline{3} = 12\,2/3$

6. Bunny is a calculus student at Factory State University, and she's having some trouble. Bunny says "This is a disaster! I'm so totally gonna fail calculus, and Daddy's gonna take away my clothes allowance! Every time I think I've got it, it turns out I'm totally wrong. There was this problem on the test about finding the volume of the top six inches of a sphere with radius 12 inches, right? So I said easy, I know the formula for the volume of the whole sphere, and I'll just divide that by 4 to get the top quarter of it, right? But they gave me no credit at all for it. How am I gonna buy those new shoes?"

Explain clearly to Bunny why what she did does or doesn't work, and how she should think about the problem.

The way Bunny did the problem will not work. The radius of the circle may be 12 inches through the origin, but it changes as you move up the circle. The radius will get smaller and smaller in the top 6 inches then say the second 6 inches. The circle isn't split into 4 equal parts; thus, by just dividing by four won't give a correct answer. *Exactly!*

She should think about the problem as one large circle being made up of lots & lots of little slices. She can then find an equation that will fit the varying radius of the slices. Once she has obtained that she should go ahead & determine the area & volume of a single slice. Then, to find the total volume she should set up an integral. To determine the limits of integration she should think of the circle as having an x & y axis, crossing it directly in the center, with 12 being the max y-value () She can then take the integral from 6 to 12. This will give her the correct volume.

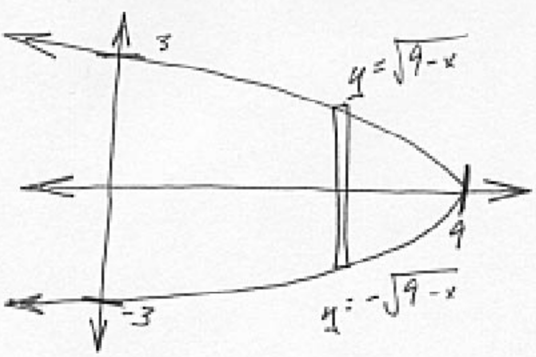
Absolutely an ideal answer.

7. **Set up an integral** for the arc length of the curve $y = 1/x$ from the point $(2, \frac{1}{2})$ to the point $(5, \frac{1}{5})$.

$$y' = \frac{-1}{x^2}$$

$$\text{Arc Length} = \int_2^5 \sqrt{1 + \left(\frac{-1}{x^2}\right)^2}\,dx$$

8. Jon has a piece of cardboard cut in the shape of the region between $x = 9 - y^2$ and the y axis. Set up integrals and use them to find $\overline{x}$, the x coordinate of the center of mass of the cardboard.



Height of a slice $= \sqrt{9-x} - -\sqrt{9-x} = 2\sqrt{9-x}$

This serves as our $\delta(x)$

$$\overline{x} = \frac{\int_0^9 x \cdot 2\sqrt{9-x}\,dx}{\int_0^9 2\sqrt{9-x}\,dx}$$

let $u = 9-x$
$du = -dx$
$-du = dx$
$x = 9-u$

$$= \frac{+\int_0^9 (9-u)\,u^{1/2}\,du}{+\int_0^9 u^{1/2}\,du}$$

$$= \frac{\int_{x=0}^{x=9} \left(9u^{1/2} - u^{3/2}\right)du}{\frac{2}{3}u^{3/2}\Big|_{x=0}^{x=9}}$$

$$= \frac{\left[9\cdot\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_{x=0}^{x=9}}{\frac{2}{3}(9-x)^{3/2}\Big|_0^9}$$

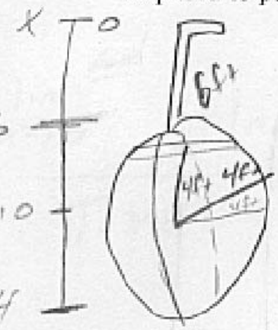
$$= \frac{\left[6(9-x)^{3/2} - \frac{2}{5}(9-x)^{5/2}\right]_0^9}{\frac{2}{3}\cdot 0^{3/2} - \frac{2}{3}\cdot 9^{3/2}}$$

$$= \frac{\left(6\cdot 0^{3/2} - \frac{2}{5}\cdot 0^{5/2}\right) - \left(6\cdot 9^{3/2} - \frac{2}{5}\cdot 9^{5/2}\right)}{-\frac{2}{3}\cdot 27}$$

$$= \frac{-6\cdot 27 + \frac{2}{5}\cdot 243}{-18}$$

$$= \frac{18}{5} \text{ or } 3.6$$

9. A spherical tank with a radius of 4 feet is buried so that its top is 6 feet underground. If the tank is full of water (with a density of 62.4 pounds per cubic foot), **write an integral** for the amount of work required to pump half the water up to the surface.



$x^2+y^2=16$

$x=\sqrt{16-y^2}$

radius $=\sqrt{16-y^2}$

area of slice $=\pi r^2 = \pi\left(\sqrt{16-y^2}\right)^2$ ft^2

volume of slice $= \pi(16-y^2)\,\Delta y$ ft^3

wt of slice $= \pi(16-y^2)\Delta y$ ft^3 $\cdot$ 62.4 lbs/ft^3

$= 62.4\pi(16-y^2)\Delta y$ lbs

$W = F\cdot d$

$62.4\pi(16-y^2)\Delta y \cdot x$

$$\boxed{\int_0^4 62.4\pi(16-y^2)(10-y)\,dy}$$

Well done

when

x	y
6	4
10	0
14	4

$m=\dfrac{4-0}{6-10}=\dfrac{4}{-4}=-1$

$y=-1x+b$

$4=-(6)+b$

$b=10 \quad y=-x+10$

$x=-y+10$

10. **Evaluate** $\int_0^1 \ln x\, dx$. $= \lim_{b \to 0} \int_b^1 \ln x\, dx$

$= \lim_{b \to 0} \left[x \ln x - x \right]_b^1$

$= \lim_{b \to 0} \left[(1 \cdot \ln 1 - 1) - (b \ln b - b) \right]$ This is a $0 \cdot \infty$ form, so rewrite it...

$= -1 - \lim_{b \to 0} b \ln b + 0$ Now it's an $\frac{\infty}{\infty}$ form, so use L'Hôpital's...

$= -1 - \lim_{b \to 0} \frac{\ln b}{1/b}$

$\overset{L'H}{=} -1 - \lim_{b \to 0} \frac{1/b}{-1/b^2}$

$= -1 - \lim_{b \to 0} -b$

$= -1$