Exam 2b Calc 2 2/20/2004

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. **Write an integral** for the area of the region between the curve $y = 9 - x^2$ and the x axis.

$y = 9 - x^2$ $\quad$ x axis : $y = 0$

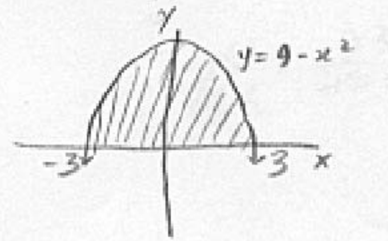
The two function meet at

$9 - x^2 = 0$

$x^2 = 9$

$x = \pm 3$

$\therefore$ at $(0, 3)$ and $(0, -3)$



Excellent!

$\therefore$ Area between $y = 9 - x^2$ and $y = 0$: $\int_{-3}^{3} (9 - x^2)\,dx$ Ans

2. If an investment pays interest at a continuous rate of 6%, and you deposit money at a constant rate of \$5000 per year, **write an integral** that gives the amount of money in the investment when you move to the Carribean to retire 20 years from now.

$r = 6\%$

$P(t) = 5{,}000$

Future Value $= \int_0^{20} 5{,}000\, e^{.06(20 - t)}\, dt$

Good

3. A spring with natural length 9 inches requires 10 foot-pounds of work to stretch it from a length of 9 inches to 12 inches. **How much work** is done in stretching the spring from 12 inches to a length of 18 inches?

$$10 \text{ ft-lbs.} = \int_0^{.25} kx$$

$$10 \text{ ft-lbs.} = \frac{1}{2} k x^2 \Big|_0^{.25}$$

$$\frac{1}{2} k \cdot \frac{1}{16} \rightarrow \frac{1}{32} k = 10 \text{ ft-lbs.}$$

$$k = 320$$

$$W = \int_{.25}^{.75} 320x$$

$$320 \cdot \frac{1}{2} x^2 \Big|_{.25}^{.75} = 160(.75)^2 - 160(.25)^2 = 80 \text{ foot-pounds}$$

$$\boxed{W = 80 \text{ ft-lbs}}$$

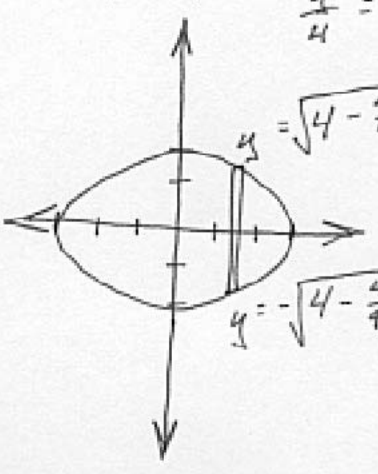
Great!

4. **Set up integrals** for $\overline{x}$, the x coordinate of the center of mass of the ellipse with equation $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

$$\frac{y^2}{4} = 1 - \frac{x^2}{9}$$

$$y = \sqrt{4 - \frac{4}{9}x^2}$$

$$y = -\sqrt{4 - \frac{4}{9}x^2}$$



$$\overline{x} = \frac{\int_{-3}^{3} x \cdot 2 \cdot \sqrt{4 - \frac{4}{9}x^2}\, dx}{\int_{-3}^{3} 2\sqrt{4 - \frac{4}{9}x^2}\, dx}$$

5. Suppose the probability density function for the length of time it takes a student to finish a math test is given by $p(x)=\begin{cases} bx^3 & \text{if } 0<x<1 \\ 0 & \text{otherwise} \end{cases}$.

(a) Find the exact value of b.

(b) Find the mean time it takes a student to complete the test.

(a) $\int_0^1 p(x) = 1$

or, $\int_0^1 bx^3 = 1$

or, $b\left[\frac{x^4}{4}\right]_0^1 = 1$

or, $b \times \frac{1}{4} = 1$

$\therefore b = 4$

b) Mean time is given by

$\int_0^1 x p(x)\, dx$

$= \int_0^1 x . 4x^3\, dx$

$= \int_0^1 4x^4\, dx$

$= 4 \cdot \left[\frac{x^5}{5}\right]_0^1$

$= 4 \times \frac{1}{5} = \frac{4}{5} = 0.8$ hrs

Well done

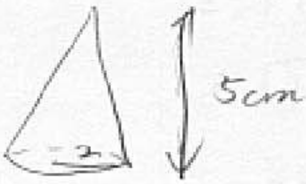
6. Set up an integral for the volume of a cone with height 5cm and base diameter 4cm, and use it to calculate the volume of the cone to the nearest cubic centimeter.

$\frac{0-2}{5-0} = -\frac{2}{5}$

$(0,2)$

$(5,0)$

$\frac{2}{5}h$ = radius



area $= \pi r^2$

$= \pi \left(\frac{2}{5}h\right)^2$

area $= \frac{4\pi}{25}h^2$

volume of a slice $= \frac{4\pi}{25}h^2 \, \Delta h$

total volume $= \int_0^5 \frac{4\pi}{25}h^2 \, dh$

$= \frac{4\pi}{25} \int_0^5 h^2 \, dh$

$= \frac{4\pi}{25} \left(\frac{h^3}{3}\right)\Big|_0^5$

$= \frac{4\pi}{75} \cdot (h^3)_0^5$

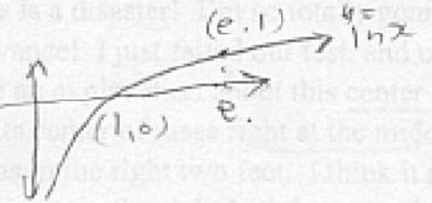
$= \frac{4\pi}{75} \cdot (125 - 0)$

Great Job

$= \frac{500\pi}{75} \approx 20.9 \text{ cm}^3$

$= \boxed{21 \text{ cm}^3}$

7. **Set up an integral** for the length of the curve $y = \ln x$ from the point $(1,0)$ to the point $(e,1)$.



$$\text{arc length} = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

$$y = \ln x$$

$$f'(x) = \frac{1}{x}$$

Perfect

$$\text{length} = \int_1^e \sqrt{1 + \frac{1}{x^2}}\, dx$$

8. Bunny is a calculus student at Factory State University, and she's having some trouble. Bunny says "This is a disaster! I'm so totally gonna fail calculus, and Daddy's gonna take away my clothes allowance! I just failed our test, and unless I can get this extra credit I'm doomed. We're supposed to write an explanation about this center of mass stuff, like if somebody said that a rod that was 4 feet long had its center of mass right at the middle, does that mean it had to have the same weight in the left two feet as in the right two feet. I think it must be true, right, because that's the only way it would balance is if the part on the right had the same weight as the part on the left, isn't it?"

Explain clearly to Bunny whether it's true or false that a rod which balances at its midpoint must have the same amount of weight on its left and right, and why.

Okay Bunny, calm down and focus on math for a minute instead of clothes. You can understand this if you try. Think about something like a baseball bat, so it's heavier at one end than the other, right? Now that way it doesn't balance in the middle, but you could add a weight on the little end so that it did balance in the middle. And you could make it balance in the middle lots of ways, really. A little bit more weight all the way out at the small end would do it, but lots more weight closer to the center could also do it - remember that stuff about "moments" from class, and how it's the product of the weight times where it is that really matters?

So there are lots of different weights we could add to the right-hand side of the bat to make it balance, depending on just how far from the center we put them. And that means there's no way you could say if the weights on the two sides are equal or not. It's really the moments that have to be equal, not the weights.

9. A spherical tank with a radius of 4 feet is buried so that its top is 6 feet underground. If the tank is half full of water (with a density of 62.4 pounds per cubic foot), **write an integral** for the amount of work required to pump the water up to the surface.

$x^2 + y^2 = 4^2$

$x = \pm\sqrt{16 - y^2}$

Radius of a slice $= \sqrt{16 - y^2}$ ft

Area of a slice $= \pi(16 - y^2)$ ft^2

Vol. of a slice $= \pi(16 - y^2)\Delta y$ ft^3

Force for a slice $= \pi(16 - y^2)\Delta y\, ft^3 \cdot 62.4\ \text{lb}/ft^3$

Work for a slice $= 62.4\pi(16 - y^2)$ lb $\cdot (10 - y)$ ft

Total Work $= \int_{-4}^{0} 62.4\pi(16 - y^2)(10 - y)\,dy$

10. **Evaluate** $\int_1^\infty \frac{\ln x}{x^2}\,dx$.

$u = \ln x \quad v = -x^{-1} = -\frac{1}{x}$

$u' = \frac{1}{x} \quad v' = x^{-2}$

Yes! +3

$= \lim_{b\to\infty} \int_1^b \left(\frac{\ln x}{x^2}\right) dx = \lim_{b\to\infty} \int_1^b \ln x (x^{-2})\,dx$

x^{-2}

$= \lim_{b\to\infty} \left[-\ln x\left(\frac{1}{x}\right) - \int_1^b \frac{1}{x}\left(-\frac{1}{x}\right) dx \right]$

Good! +4

$= \lim_{b\to\infty} \left[-\frac{\ln x}{x} - \frac{1}{x} \right]_1^b = \lim_{b\to\infty} -\frac{\ln b}{b} - \frac{1}{b} - \left(\frac{-\ln|1|}{1} - \frac{1}{1}\right)$

$\lim_{b\to\infty} -\frac{\ln b}{b} \overset{L'H}{=} \frac{-\frac{1}{b}}{1} = \frac{0}{1}$

$= 0 - 0 + 0 + 1 = 1$

Wonderful!! +3

Nice Job!