Exam 3a Calc 2 3/26/2004

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Find the sum of the series $1-\frac{2}{5}+\frac{4}{25}-\frac{8}{125}+\ldots$

$\left(-\frac{2}{5}\right) \quad \left(-\frac{2}{5}\right)$

geometric series $a = 1$

$r = -\frac{2}{5}$

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1+\frac{2}{5}} = \frac{1}{7/5} \quad \boxed{\frac{5}{7}}$$

Good

2. Write the series $(x-1)-\frac{(x-1)^2}{2!}+\frac{(x-1)^3}{3!}-\frac{(x-1)^4}{4!}+\frac{(x-1)^5}{5!}-\ldots$ in sigma notation.

$$\boxed{\sum_{n=1}^{\infty} \frac{(x-1)^n}{n!}(-1)^{n+1}}$$

check $\frac{(x-1)}{1} - \frac{(x-1)^2}{2!} + \ldots$ ✓

Great!

3. Write a 7^{th} degree Taylor polynomial for the function $f(x) = x^2 \sin(2x)$ centered at zero.

We know,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$\sin(2x) = (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!}$$

$$x^2 \cdot \sin(2x) = x^2\left[(2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!}\right]$$

$$= 2x^3 - \frac{2^3 \cdot x^5}{3!} + \frac{2^5 x^7}{5!} - \frac{2^7 x^9}{7!}$$

Excellent

$\therefore$ The 7^{th} degree polynomial will be

$$x^2 \cdot \sin(2x) = 2x^3 - \frac{2^3 \cdot x^5}{3!} + \frac{2^5 x^7}{5!} \;//$$

4. Several derivatives of the function $f(x) = \sec x$ are given below. Use them to find the 3rd degree Taylor polynomial for $\sec x$ centered at $x = 0$.

$f'(x) = \sec x \tan x$

$f''(x) = \sec x \tan^2 x + \sec x$

$f^{(3)}(x) = \sec x \tan^3 x + 5\sec^3 x \tan x$

$f(0)$ $\sec x = \frac{1}{\cos x} = \underline{1}$

$f'(0)$ $\sec x \tan x = \underline{0}$

$f''(0)$ $\sec x \tan^2 x + \sec x = \underline{1}$

f $= \underline{0}$

$$= \boxed{1 + 0x + \frac{1}{2!}x^2 + \frac{0}{3!}x^3}$$

Great

$\sec(0) = 1$

$\tan(0) = 0$

put 0 into the derivative equations that gives us the constant terms to put into the Taylor polynomial

5. Determine whether the series $\sum_{n=0}^{\infty} \frac{1}{n^3+1}$ converges or diverges.

Use comparison test.

$n^3 < n^3 + 1$

$\frac{1}{n^3} > \frac{1}{n^3+1}$ (bigger denominator smaller #)

So... we know $\sum_{n=0}^{\infty} \frac{1}{n^3}$ will converge because of the p series test, and $3>1$.

so if $\sum_{n=0}^{\infty} \frac{1}{n^3}$ converges so does $\sum_{n=0}^{\infty} \frac{1}{n^3+1}$

Wonderful

6. Is x = -1 included in the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$?

to check this: put -1 in for x into ↑ and see what happens. Ok, here goes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

By AST: $a_n = \frac{1}{n}$

(1) is a_n decreasing?

$\frac{1}{n} > \frac{1}{n+1}$ yes, the denom. is increasing so the # itself is getting smaller.

(2) is $\lim_{n \to \infty} \frac{1}{n} \stackrel{?}{=} 0$

$\frac{1}{\infty} = 0$ yes

if (1) & (2) are true then this series converges by the AST

this series converges when $x = -1$ so it

should be included in the interval of convergence

Well done

7. What is the radius of convergence of the series $\sum_{n=0}^{\infty}(3x)^n$?

$$\sum_{n=0}^{\infty}(3x)^n$$

Using ratio test,

$$\lim_{n\to\infty}\left|\frac{(3x)^{n+1}}{(3x)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\cancel{(3x)^n}\cdot 3x}{\cancel{(3x)^n}}\right|$$

$$= |3x|$$

$$|3x| < 1.$$

$$|x| < 1/3$$

Excellent

$\therefore$ Radius of convergence is $\underline{1/3}$.

8. Biff is having calculus trouble again. He says "Dude, this series stuff is kicking my ass. I was talking to this friend of mine, and his calculus class uses a different book because he's an engineering major or something, and he was saying there's this other test for whether one converges or not. It's like, you take the square root of it and see if that converges or not, and that tells you if yours converges or not. That seems pretty cool to me, since ~~I was pretty good at math~~ back when it was just doing square roots and stuff."

Tell Biff whether you think he should believe what his friend told him or not, and why. Be clear and justify your answer, keeping it in terms Biff can understand.

Dear Biff, I think your friend is wrong. There is no such test to see whether the series converge or not. Well, see if you don't believe me I will give you an example. Let us take a series $\sum_{n=0}^{\infty} \frac{1}{n^2}$, which we already know that it converges. Now, lets do as your friend said,

$$\sum_{n=0}^{\infty} \sqrt{\frac{1}{n^2}} = \sum_{n=0}^{\infty} \frac{1}{n}$$

Excellent use of a choice example.

We know, $\sum_{n=0}^{\infty} \frac{1}{n}$ is a mono-harmonic series and it diverges. Then if we consider your friend was right then $\sum_{n=0}^{\infty} \frac{1}{n^2}$ should diverge. But in real $\sum_{n=0}^{\infty} \frac{1}{n^2}$ converges.

Biff, I think you understood. If you have any other question please feel free to ask.

9. For which values of p will the series $\sum_{n=0}^{\infty} \frac{e^n}{\left(e^n+1\right)^p}$ converge?

Let's use integral test. Good choice.

$$\int_0^\infty \frac{e^n}{(e^n+1)^p}\,dn \qquad \text{Let } u = e^n+1$$

$$\frac{du}{dn} = e^n$$

$$\lim_{b\to\infty} \int_0^b \frac{\cancel{e^n}}{u^p}\cdot\frac{du}{\cancel{e^n}} \qquad \therefore dn = \frac{du}{e^n}$$

$$= \lim_{b\to\infty} \int_0^b \frac{1}{u^p}\,du$$

$$= \lim_{b\to\infty} \left[\frac{u^{-p+1}}{-p+1}\right]_0^b = \lim_{b\to\infty}\left[\frac{(e^n+1)^{-p+1}}{-p+1}\right]_0^b$$

The above value will diverge if $p=1$ and less than 1.

Thus the integral converge at $P>1$

This is integral test so, same applies to the series. The series converges at $P>1$

Beautifully done.

10. Differentiate the Taylor series for xe^x and use the result to show that $\sum_{n=0}^{\infty} \frac{n+1}{n!} = 2e$.

We know

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$x \cdot e^x = x\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right]$$

$$x \cdot e^x = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \cdots$$

$$\frac{d}{dx}(x \cdot e^x) = \frac{d}{dx}\left[x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \cdots\right]$$

$$e^x(1+x) = 1 + 2x + \frac{3x^2}{2!} + \frac{4x^3}{3!} + \frac{5x^4}{4!}$$

$$e^x(1+x) = \sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!}$$

$$\sum_{n=0}^{\infty} \frac{n+1}{n!} = 1 + 2 + \frac{3}{2!} + \frac{4}{3!} + \frac{5}{4!}$$

Yes!

$e^x(1+x) = 2e$, when $x = 1$

$$\frac{d}{dx} xe^x = 1 \cdot e^x + x \cdot e^x = e^x(1+x)$$

$$\therefore e^1(1+1) = \sum_{n=0}^{\infty} \frac{(n+1)1^n}{n!}$$

$$\boxed{2e = \sum_{n=0}^{\infty} \frac{n+1}{n!}}$$

Well done