Exam 4b Calc 2 4/23/2004

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Determine whether $y = 3e^{-2t}$ is a solution to the differential equation $\dfrac{d^2y}{dt^2} = 9y$.

$y = 3e^{-2t}$

$y' = -2 \cdot 3e^{-2t} = -6e^{-2t}$

$y'' = -2 \cdot -6e^{-2t} = 12e^{-2t}$

$\dfrac{d^2y}{dt^2} = 9y$

Good

$12e^{-2t} \overset{?}{=} 9(3e^{-2t})$

$12e^{-2t} \overset{?}{=} 27e^{-2t}$

No, $y = 3e^{-2t}$ is not a solution

2. Find a general solution to the differential equation $\dfrac{dy}{dx} + xy^3 = 0$.

sep. of var. $\dfrac{dy}{dx} = -xy^3$

int. $\displaystyle\int \frac{1}{y^3}\,dy = \int -x\,dx$

$\displaystyle\int y^{-3}\,dy = -\frac{x^2}{2} + C$

$-\frac{1}{2}y^{-2} = -\frac{x^2}{2} + C$

$\dfrac{y^{-2}}{2} = \dfrac{x^2}{2} - C$

$y^{-2} = x^2 - 2C$

$\dfrac{1}{y^2} = x^2 - 2C$

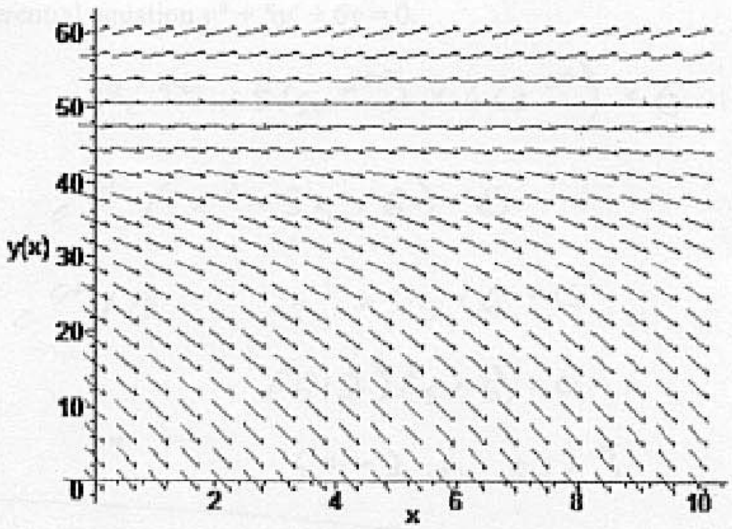
$y^2 = \dfrac{1}{x^2 - 2C}$

Great

Yes! :)

$\boxed{y = \pm\sqrt{\dfrac{1}{x^2 - 2C}}}$

3. Which of the differential equations could have the slope field pictured at right?



a) $\dfrac{dp}{dt} = 0.2p$

b) $\dfrac{dp}{dt} = -0.2p$

c) $\dfrac{dp}{dt} = 0.2p + 20$

d) $\dfrac{dp}{dt} = 0.2p - 10$

e) $\dfrac{dp}{dt} = 0.2p(50 - p)$

a would always be increasing, as would c

b would always be decreasing

d has an equilibrium at 50, as does e. e would also have an equilibrium at 0 since the lines on the x axis are still pointing down, so assuming there is no equilibrium at 0.

So, d is the one that fits, because it only has 1 equilibrium at 50, the same as the slope field

Excellent!

4. Find a general solution for the differential equation $y'' + 5y' + 6y = 0$.

Suppose $y = e^{st}$ is a solution.

$\therefore\ y' = se^{st}$

$y'' = s^2 e^{st}$

Substituting:

$(s^2 e^{st}) + 5(se^{st}) + 6(e^{st}) = 0$

or, $e^{st}(s^2 + 5s + 6) = 0$

or, $(s^2 + 3s + 2s + 6) = 0 \qquad (\because e^{st} \neq 0)$

or, $s(s+3) + 2(s+3) = 0$

or, $(s+3)(s+2) = 0$

$\therefore\ s = -3$ or $s = -2$

Hence,

$y = ae^{-3t} + be^{-2t}$ is the general solution

Good

5. Find all equilibrium points of the system

$$\frac{dx}{dt} = x - x^2 - \frac{xy}{3}$$

$$\frac{dy}{dt} = y - y^2 - \frac{xy}{2}$$

At equilibrium, have no change

$\frac{dx}{dt} = x - x^2 - \frac{xy}{3} \Rightarrow x = 0$ or $x - x^2 - \frac{xy}{3} = 0 \Rightarrow \frac{1}{x}(x - x^2) = \frac{xy}{3} \cdot \frac{1}{x}$

$\Downarrow$

$\frac{dy}{dt} = y - y^2 - \frac{xy}{2}$

$y - y^2$

$y(1-y)$

$y = 0$ or $y = 1$

$\frac{x - x^2}{x} = \frac{y}{3}$

$1 - x = \frac{y}{3}$

$-x = \frac{y}{3} - 1$

$x = -\frac{y}{3} + 1$

Put back in here to find x

$0 = y - y^2 - \frac{(-\frac{y}{3} + 1)y}{2}$

$0 = y - y^2 - \frac{(-\frac{y^2}{3} + y)}{2}$

$0 = y - y^2 + \left(\frac{\frac{y^2}{3} - y}{2}\right)$

$0 = y - y^2 + \frac{1}{6}y^2 - \frac{y}{2}$

$0 = \frac{1}{2}y + \frac{-5}{6}y^2$

$0 = y\left(\frac{1}{2} + \frac{-5}{6}y\right)$

$y = 0$ or $y = 3/5$

$(0,0), (0,1), (1,0)\ \left(\frac{4}{5}, \frac{3}{5}\right)$

Excellent

$-\frac{5}{6}y = -\frac{1}{2}$

$-5y = -3$

$y = 3/5$

6. Suppose a 70° kumquat is placed in an oven that begins at 70° but which heats linearly to 350° over the next 10 minutes. Write a differential equation fo rthe temperature of the kumquat after t minutes and use Euler's method with $\Delta t = 5$ minutes to estimate the temperature (correct to 2 decimal places) of the kumquat at the end of 10 minutes.

$$\frac{dH}{dt} = k\left((70+28t) - H\right)$$

$$\frac{dH}{dt} = .0673\left[(70+28\,t) - H\right]$$

(t, H)	ΔH	$H(t+\Delta t)$
$(0, 70)$	0	70
$(5, 70)$	47.11	117.11
$\boxed{(10, 117.11)}$		

for $t=0$: $\frac{dH}{dt} = .0673\left[(70+28\cdot 0) - 70\right]$

$\frac{dH}{dt} = .0673(0)$

$dH = 0$

for $t=5$

$\frac{dH}{dt} = .0673\left[(70+28\cdot 5) - 70\right]$

$\frac{dH}{dt} = .0673(140)$

$\frac{dH}{dt} = 9.422$

$dt = 5$

so $dH = 9.422\cdot dt$

$dH = 9.422\cdot 5$

$dH = 47.11$

after 10 minutes, the kumquat will be 117.11°

Great

7. Bunny is having trouble with differential equations. She says "Ohmygod! It's so totally unfair! I mean, I can do math okay when they give the directions right, you know? But now they totally just don't tell us what to do, and I'm lost. They said to pick a suitable kind of differential equation for how many people have cable modems in their houses and use it, but that's so wrong, because how am I supposed to know? I mean, when they say something's proportional to something I can do that, but how am I supposed to know what's proportional to cable modems?"

Suggest to Bunny what sort of model might be appropriate for this situation, and why.

Bunny needs to use a logistic growth equation to limit the growth. Good call. She needs to either set it up as a percent of the total population, or as # of people. If she uses the # of people with cable modems, the equation would look something like:

$$\frac{dC}{dt} = k(75 \text{ million} - C)$$

, where C is the number of people with cable modems, & k is a constant. The 75 million comes from approximately 300 million people in the country, with an average family size of 4, thus giving us approximately 75 million houses in the US. As the number of people with cable modems increases the growth rate will slow, until (theoretically) every house has one, and the growth rate will become zero.

Very nice answer!

8. Lake Erie has a volume of 460,000 km³ and an outflow rate of 175 km³ per year. Suppose that 20kg of a certain pollutant is present in the lake at time 0. How long (to the nearest year) will it be until only 5kg remain?

$$k = -\frac{r}{V} = -\frac{175}{460{,}000} = -.0003804$$

$$\frac{dQ}{dt} = k \cdot Q$$

$$Q = Ae^{kt} \qquad Q = Q_0 e^{kt}$$

$$Q = Q_0 e^{-.0003804t}$$

$$Q_0 = Ae^{0}$$

$$Q_0 = A$$

we want t when $Q = .25$ of Q_0

so

$$.25 Q_0 = Q_0 e^{-.0003804t}$$

$$.25 = e^{-.0003804t}$$

$$\ln .25 = -.0003804 \cdot t$$

$$-\frac{\ln .25}{.0003804} = t$$

Excellent.

$\boxed{3{,}644 \text{ years} = t}$

seems a little off...

my k must be off

Actually my value was off!

9. Suppose that the population of carp in a certain river grows logistically with a carrying capacity of 8000 fish, and that when there are 6000 fish in the river the population grows at a rate of 480 fish per year. How many fish can safely be harvested from the river without causing extinction?

$$\frac{dP}{dt} = k \cdot p(8000 - p)$$

When $p = 6000$, $\frac{dP}{dt} = 480$, so

$$(480) = k \cdot (6000)(8000 - (6000))$$

$$k = .00004$$

So $\frac{dp}{dt} = .00004p(8000 - p)$

Since I know max growth at $L/2$, or $p = 4000$:

$$\frac{dp}{dt} = .00004(4000)(8000 - (4000)) = 640$$

So as long as the population begins above 4000, it should be safe to harvest up to 640 fish – as long as conditions remain the same.

10. The differential equation $y'' + 4y' = 3\cos 5t$ isn't quite the sort where we can use our standard characteristic polynomial strategy, but it's close. Suppose that there's a solution of the form $y = a\cos 5t$ for some value of the constant a, and see if you can find a workable value for a.

$y = a\cos 5t$

$y' = -5a\sin 5t$

$y'' = -25a\cos 5t$

Now, $-25a\cos 5t + 4a\cos 5t = 3\cos 5t$

or, $\cos 5t(-25a + 4a) = 3\cos 5t$

or, $-21a\cos 5t = 3\cos 5t$

$\therefore$ $-21a$ should be equal to 3 from the above eqn

or, $-21a = 3$

$\therefore a = \frac{-1}{7}$

Nice Job