Each problem is worth 10 points. For full credit indicate clearly how you reached your answer.

1. Verify that $y = e^{2t}$ is a solution to the differential equation $y'' - 5y' + 6y = 0$.

$$y' = 2e^{2t}$$
$$y'' = 4e^{2t}$$

$$4e^{2t} - 5(2e^{2t}) + 6(e^{2t})$$
$$4e^{2t} - 10e^{2t} + 6e^{2t} = 0.$$

Great

2. Give examples of (make clear which is which!):

(a) An autonomous differential equation $\frac{dy}{dt}$ only depends on y

$$\frac{dy}{dt} = y$$

Exactly

(b) A non-autonomous differential equation

$$\frac{dy}{dt} = y^t$$

$\frac{dy}{dt}$ depends on t

CO = amount of carbon monoxide

At $3Am$ total $CO = 2000(45) = 90000 = CO$

3. Jon's bedroom (which has a volume of roughly 2000 cubic feet) reached a peak of 45 parts per million of carbon monoxide at 3am, at which point he opened the windows. If 5 cubic feet of clean air (with zero parts per million of carbon monoxide) entered through one window each minute and 5 cubic feet of air from the room exited through the other window each minute, **write a differential equation** for the carbon monoxide level in the room t minutes after 3am.

$$\frac{d(CO)}{dt} = 5(0) - \frac{5(CO)}{2000}$$

clean air incoming   bad air outgoing
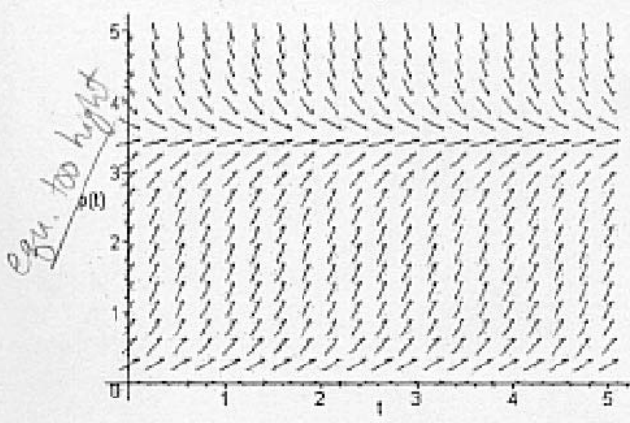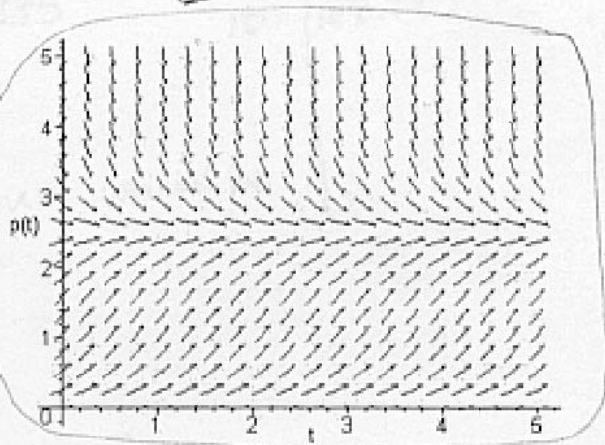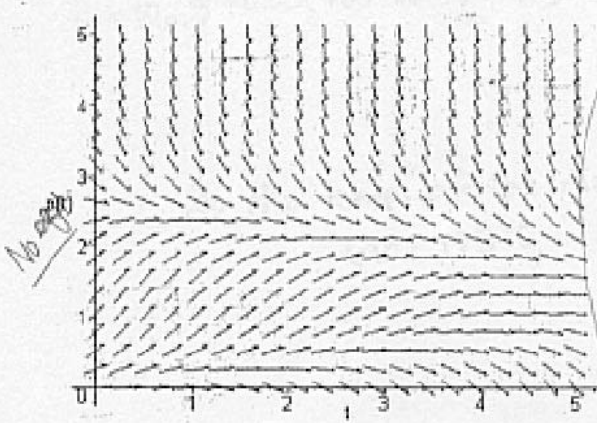
units of $\frac{d(CO)}{dt} = \frac{(cuft)(ppm)}{(cu.ft)}$

CO level drops after 3Am.

$$CO(0) = 45$$ initial condition.

$45_{ppm} @ 3Am$
$+ 0_{ppm} @ 5cu.ft/min$ — room air @ 5cu.ft/min.

Excellent

4. Which of the slope fields pictured below could represent the differential equation

$$\frac{dP}{dt} = -P(P - 2.5)?$$ equ @ $P = 0$ & 2.5 yes!

No equ

equ too high

No equ @0

5. If the differential equation $\dfrac{dT}{dt} = 0.05(400 - T)$ represents the temperature after t minutes of a toy mouse (which starts out at 70°) that Jon's cat Nemo drops into the oven, **use Euler's method** with $\Delta t = 5$ to estimate the temperature of the mouse ten minutes later when Jon notices that something smells funny.

$$T(0) = 70°$$

at $T(5)$:

$$T(5) = T(0) + \frac{dT}{dt} \Delta t$$

$$= 70 + \left(.05(400 - 70) \right)(5)$$

$$= 70 + 82.5$$

$$= \underline{152.5}$$

at $T(10)$

$$T(10) = T(5) + \frac{dT}{dt} \Delta t$$

$$= 152.5 + \left(.05(400 - 152.5) \right)(5)$$

$$= 152.5 + 61.875$$

$$= \underline{214.375}$$

Temp at $t = 10$ min $\approx 214.4°$

Nice Job

6. **Find the general solution to the differential equation** $\dfrac{dy}{dt} = \dfrac{t}{t^2 y + y}$

$$\frac{dy}{dt} = \frac{1}{y}\left(\frac{t}{t^2+1}\right)$$

$$y\,dy = \left(\frac{t}{t^2+1}\right)dt$$

$$\int y\,dy = \int \frac{t}{t^2+1}\,dt$$

$$\frac{y^2}{2} = \frac{1}{2}\ln(t^2+1) + c$$

$$y^2 = \ln(t^2+1) + C \qquad \text{Excellent}$$

$$y = \pm\sqrt{\ln(t^2+1) + C}$$

let $u = t^2+1$

$du/dt = 2t$

$dt = \dfrac{du}{2t}$

$$\int \frac{t}{t^2+1}dt = \int \frac{t}{u}\frac{du}{2t} =$$

$$\frac{1}{2}\int \frac{1}{u}\,du$$

$$\frac{1}{2}\ln|u| + c$$

$$\frac{1}{2}\ln|t^2+1| + C$$

t^2+1 is always positive

so $\frac{1}{2}|\ln|t^2+1||^c = \frac{1}{2}\ln(t^2+1)^+C$

7. Jon's friend Chris went to grad school in an environmental studies program. Chris had a graduate level course where they did remarkably simplistic modeling of populations over time. He got a vague idea there that sometimes you could use differential equations with quadratic terms in them to represent interactions, so he's tried to make a model for the population of elephants in a game preserve in Africa. Since the quadratic term represents interactions, Chris thinks it should have a positive coefficient (more meetings between boy elephants and girl elephants leads to more baby elephants, right?). He's very confused by the slope fields he's getting from his model, and thinks something might be wrong. **Explain** clearly and in simple terms to Chris whether a quadratic function with positive leading coefficient is well suited to modeling population growth of elephants, and why.

A quadratic function with a positive leading coefficient is <u>not</u> well suited for modeling population growth of elephants because, as you can see from your slope fields, there is no cap on the growth of the population. The quadratic form can be simplified into a logistic problem which has a parameter called the carrying capacity. (i.e. $\frac{dP}{dt} = P(1 - \frac{P}{N})$). This value is a parameter that is usually determined by how many elephants the environment can support. If the population exceeds this value it will decline to hit this equilibrium. If the leading coefficient is positive, this value is no longer a sink and the population will continue to grow forever (according to the slope field)

Excellent analysis.

8. Find a particular solution to the differential equation $\dfrac{dy}{dt} = e^{2t} + 3y$ subject to the initial condition $y(0) = 3$.

$$\frac{dy}{dt} - 3y = e^{2t} \qquad \underline{\text{linear}}$$

$$u(t) = e^{\int -3 dt}$$

$$u(t) = e^{-3t}$$

$$e^{-3t}\frac{dy}{dt} - 3e^{-3t}y = e^{-3t}e^{2t}$$

$$\frac{d}{dt}\left(ye^{-3t}\right) = e^{-t}$$

$$ye^{-3t} = -e^{-t}$$

$$y = -e^{2t} + Ce^{3t}$$

$$3 = -1 + C$$

$$C = 4$$

Well done

$$\boxed{y = -e^{2t} + 4e^{3t}}$$

<u>Check!</u>

$$\frac{dy}{dt} = -2e^{2t} + 12e^{3t}$$

$$-2e^{2t} + 12e^{3t} \stackrel{?}{=} e^{2t} + 3(-e^{2t} + 4e^{3t})$$

$$-2e^{2t} + 12e^{3t} = -2e^{2t} + 12e^{3t}$$

9. **Sketch the bifurcation diagram** for the differential equation $\frac{dy}{dt} = y^6 - 2y^4 + \alpha$. Include direction arrows on the phase lines and make clear the exact α values where bifurcations occur.

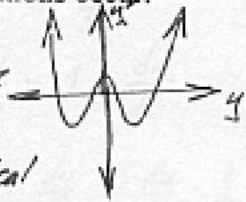Equilibrium when $0 = y^6 - 2y^4 + \alpha \rightarrow$ Graph looks like this, $\rightarrow$
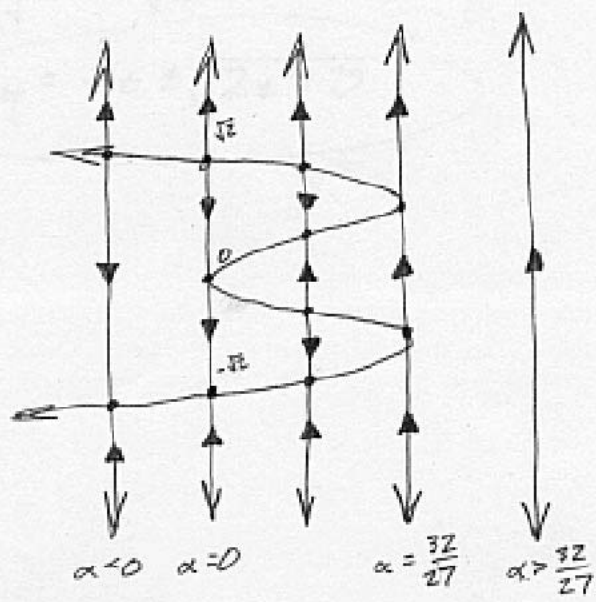
Max/min when $0 = 6y^5 - 8y^3$

$$0 = 2y^3(3y^2 - 4)$$

so when $y = 0$ or when $y = \pm \frac{2}{\sqrt{3}}$

when $y = 0$,
$\frac{dy}{dt} = 0$ also

When $y = \pm \frac{2}{\sqrt{3}}$,

$\frac{dy}{dt} = \frac{2^6}{3^3} - 2 \cdot \frac{2^4}{3^2}$

$= \frac{64}{27} - \frac{32}{9}$

$= \frac{-32}{27}$

but with a vertical translation by α that changes where and how many horizontal intercepts (or equilibria) we have. So it's a matter of how far those minima were below the axis in the basic $y^6 - 2y^4$ graph, which amounts to the Calc 1 problem at left.

$\alpha < 0 \quad \alpha = 0 \qquad\qquad \alpha = \frac{32}{27} \quad \alpha > \frac{32}{27}$

10. Find a general solution to the differential equation $(t + y)\dfrac{dy}{dt} = t - y$ [Hint: The substitution $u = t + y$ might be helpful].

$$(t + y)\frac{dy}{dt} = t - y \qquad\qquad let \quad u = t + y \qquad\qquad y = u - t$$

$$\frac{du}{dt} = 1 + \frac{dy}{dt}$$

$$(u)\left(\frac{du}{dt} - 1\right) = t - (u - t)$$

$$\left(\tfrac{1}{u}\right) u \left(\frac{du}{dt} - 1\right) = (2t - u)\left(\tfrac{1}{u}\right)$$

$$\frac{du}{dt} - 1 = \frac{2t}{u} - 1$$

$$\int u \, du = \int 2t \, dt$$

$$\tfrac{1}{2} u^2 = t^2 + C$$

$$u^2 = 2t^2 + 2C$$

$$u = \pm\sqrt{2t^2 + 2C}$$

$$t + y = \pm\sqrt{2t^2 + 2C}$$

$$\boxed{y = \pm\sqrt{2t^2 + 2C} - t}$$

Excellent