Exam 2 Differential Equations 3/5/04

Each problem is worth 10 points. For full credit indicate clearly how you reached your answer.

1. **Verify** that $y = e^{2t}$, $v = 2e^{2t}$ is a solution to the system of equations

$$\frac{dv}{dt} = -6y + 5v$$
$$\frac{dy}{dt} = v$$

$y = e^{2t}$ $\quad \frac{dy}{dt} = 2e^{2t}$

$V = 2e^{2t}$ $\quad \frac{dv}{dt} = 4e^{2t}$

$\frac{dv}{dt} = 4e^{2t} \stackrel{?}{=} -6(e^{2t}) + 5(2e^{2t}) = -6e^{2t} + 10e^{2t}$

$= 4e^{2t}$ ✓

$\frac{dy}{dt}$ $\quad 2e^{2t} \stackrel{?}{=} 2e^{2t}$ ✓

Exactly

yes, it is a solution because it works

2. State the definition of the Laplace transform of a function $y(t)$.

$\mathcal{L}[y(t)] = \int_0^\infty y(t) \cdot e^{-st}\, dt$ Yes

3. Suppose that the populations of rabbits and ferrets are governed by the differential equations

$$\frac{dR}{dt} = 2R - 1.2RF$$
$$\frac{dF}{dt} = -F + 1.2RF$$

. If the rabbit population begins at 2 and the ferret population begins at 1 (where both populations are measured in thousands), use Euler's method with step size $\Delta t = 0.5$ to find the missing value from the table below (do not round).

$R = 2 \quad F = 1$

t	R	F
0	2	1
.5	2.8	1.7
1	2.744	3.706

$\Delta t = 0.5 \quad \frac{dR}{dt} = 2(2) - 1.2(2)(1)$

$dR = 0.8$

$\frac{dF}{dt} = -1 + 1.2(2)(1)$

Nice Job

$dF = 0.7$

$\Delta t = 0.5 \quad dF = \left[-1.7 + (1.2)(2.8)(1.7)\right] 0.5$

$= 2.006$

$\Delta t = 0.5 \quad dR = \left[2(2.8) - 1.2(2.8)(1.7)\right] 0.5$

$dR = -0.056$

$F = 1.7 + 2.006$

4. Find all equilibrium points of the system
$$\frac{dx}{dt} = 5x\left(1-\frac{x}{5}\right)-xy$$
$$\frac{dy}{dt} = 3y\left(1-\frac{y}{3}\right)-2xy$$
.

$5x\left(1-\frac{x}{5}\right)-xy=0$

$x(5-x-y)=0$

$\Rightarrow \underline{x=0}$ or $x+y=5$

if $y=0 \Rightarrow x=5$

$3y\left(1-\frac{y}{3}\right)-2xy$

$y(3-y-2x)=0$

$\Rightarrow \underline{y=0}$ or $2x+y=3$

if $x=0 \Rightarrow y=3$.

$x+y=5$
$2x+y=3$
$-x=+2$
$x=-2 \Rightarrow y=7$

Well done

Eq $\underline{(0,0)}$ $\underline{(-2,7)}$ $\underline{(5,0)}$ $\underline{(0,3)}$.

5. Covert the differential equation $\frac{d^2y}{dt^2}+4\frac{dy}{dt}-12y=0$ to a system of two first order differential equations.

let $v=\frac{dy}{dt}$, so $v'=\frac{d^2y}{dt^2}=\frac{dv}{dt}$

$\frac{d^2y}{dt^2}+4\frac{dy}{dt}-12y=0$

$\frac{dv}{dt}+4v-12y=0$

Great

$$\boxed{\frac{dv}{dt}=12y-4v \quad \& \quad \frac{dy}{dt}=v}$$

6. Find a solution to the differential equation $y + y' = 3x^2 + 2x$ by assuming there is a second degree polynomial solution.

a second degree polynomial looks like

$y = ax^2 + bx + c$

$y' = 2ax + b$

Nice Job!!

so the solution looks like: $y = 3x^2 - 4x + 4$

Plug in:

$ax^2 + bx + c + 2ax + b = 3x^2 + 2x$

$ax^2 + (2a+b)x + (c+b) = 3x^2 + 2x$

Match terms:

$ax^2 = 3x^2$

$a = 3$

$2a + b = 2$

$2(3) + b = 2$

$6 + b = 2$

$b = 2 - 6$

$b = -4$

$c + b = 0$

$c + (-4) = 0$

$c = 4$

check on back

7. Prove that $L\left[\frac{dy}{dt}\right] = sL[y] - y(0)$.

$$\mathcal{L}\left[\frac{dy}{dt}\right] = \int_0^\infty \frac{dy}{dt} e^{-st}\, dt \qquad \text{using} \quad u = e^{-st} \quad v = y(t)$$
$$du = -se^{-st} \quad dv = \frac{dy}{dt}$$

$$= \lim_{b\to\infty} \int_0^b \frac{dy}{dt} e^{-st}\, dt = \lim_{b\to\infty} \left[y(t)e^{-st}\Big|_0^b - \int_0^b -s\,y(t)e^{-st}\, dt \right]$$

$$= \lim_{b\to\infty} y(b)e^{-s(b)} - y(0)e^{-s(0)} + s\int_0^b y(t)e^{-st}\, dt$$

we recognize this as the definition of $\mathcal{L}[y(t)]$

$$= \lim_{b\to\infty} \frac{y(b)}{e^{s(b)}} - \frac{y(0)}{e^0} + s\,\mathcal{L}[y(t)]$$

Absolutely Excellent!

$$= \frac{y(\infty)}{e^{s\infty}} - y(0) + s\,\mathcal{L}[y(t)]$$

* under the condition that y is a "well behaved" function then $y(t)$ will grow slower than e^{st} so $\frac{y(\infty)}{e^{s b}}$ goes to zero

$$= -y(0) + s\,\mathcal{L}[y(t)]$$

$$\mathcal{L}\left[\frac{dy}{dt}\right] = s\,\mathcal{L}[y(t)] - y(0) \quad \blacksquare$$

8. Find a general solution to the system $\dfrac{dx}{dt} = -2x + y$, $\dfrac{dy}{dt} = 3y$.

partially decoupled

so
$\frac{dy}{dt} = 3y$

guess a solution

ke^{3t} would be a solution

plug-in to $\frac{dx}{dt}$

$e^{\int 2\,dt} = e^{2t}$

linear

$$e^{2t}\left(\frac{dx}{dt} + 2x = ke^{3t}\right)$$

$$e^{2t}\cdot\frac{dx}{dt} + 2x\cdot e^{2t} = ke^{5t}$$

$$xe^{2t} = \int ke^{5t}\cdot dt$$

$$xe^{2t} = \frac{ke^{5t}}{5} + C$$

Great Job

$$x = \boxed{\frac{ke^{3t}}{5} + C\cdot e^{-2t}}$$

$$y = \boxed{ke^{3t}}$$

Check: Smart.

$$\frac{3}{5}ke^{3t} \cancel{- 2Ce^{-2t}} = \frac{-2ke^{3t}}{5} \cancel{-2Ce^{-2t}} + ke^{3t}$$

$$\frac{3}{5}ke^{3t} + \frac{2}{5}ke^{3t} = ke^{3t}$$

$$\frac{5}{5}ke^{3t} = ke^{3t}$$

$$ke^{3t} = ke^{3t}$$

9. Find a solution to the differential equation $y'' - 4y = 2e^{3t}$ by assuming there is a solution of the form $y = Ae^{3t}$.

$y = Ae^{3t}$

$y' = 3Ae^{3t}$

$y'' = 9Ae^{3t}$

$y'' - 4y = 2e^{3t}$

$9Ae^{3t} - 4(Ae^{3t}) = 2e^{3t}$

$9Ae^{3t} - 4Ae^{3t} = 2e^{3t}$

$(9A - 4A)(e^{3t}) = 2e^{3t}$

$9A - 4A = 2$

$A(9-4) = 2$

$5A = 2$

$A = \frac{2}{5}$

Very Nice

$y = \frac{2}{5}e^{3t}$ is a possible solution.

Check: $y = \frac{2}{5}e^{3t}$ $\quad y' = \frac{6}{5}e^{3t}$ $\quad y'' = \frac{18}{5}e^{3t}$

$\frac{18}{5}e^{3t} - 4\left(\frac{2}{5}e^{3t}\right) \stackrel{?}{=} 2e^{3t}$

$\left(\frac{18}{5} - \frac{8}{5}\right)(e^{3t}) \stackrel{?}{=} 2e^{3t}$

$\frac{10}{5}e^{3t} \stackrel{?}{=} 2e^{3t}$

$2e^{3t} = 2e^{3t}$ ✓ It works!

10. a) Find a general solution to the differential equation $y'' - 4y = 0$.

b) Find a solution to the differential equation $y'' - 4y = 2e^{3t}$ satisfying the initial condition $y(0) = 5$ [This takes some insight, but think about how your answers to 9. and 10. a) can be combined].

(a) $y'' - 4y = 0$

guess $y = ke^{st}$

$y'' = s^2 ke^{st}$

$s^2 ke^{st} - 4ke^{st} = 0$

$ke^{st}(s^2 - 4) = 0$

$ke^{st}(s+2)(s-2) = 0$

$s = -2$

$s = 2$

two solutions could be:

$\boxed{y = ke^{2t} \text{ or } y = ke^{-2t}}$ Yes

(b) $y'' - 4y = 2e^{3t}$ satisfying initial condition $y(0) = 5$

$y = (k_1 e^{2t} + k_2 e^{3t})$

k_1 = anything

$y'' = 4k_1 e^{2t} + 9k_2 e^{3t}$

$\cancel{4k_1 e^{2t}} + 9k_2 e^{3t} - \cancel{4k_1 e^{2t}} - 4k_2 e^{3t} = 2e^{3t}$

$5k_2 e^{3t} = 2e^{3t}$

$k_2 = \frac{2}{5}$

$y = k_1 e^{2t} + \frac{2}{5} e^{3t}$

$y(0) = 5$

$5 = k_1 e^{2(0)} + \frac{2}{5} e^{3(0)} \Rightarrow 5 = k_1 + \frac{2}{5}$ $k_1 = 5 - \frac{2}{5} = \frac{25}{5} - \frac{2}{5} = \frac{23}{5}$

so, $\boxed{y = \frac{23}{5} e^{2t} + \frac{2}{5} e^{3t}}$

Check:

$y'' = \frac{92}{5} e^{2t} + \frac{18}{5} e^{3t}$

$\frac{92}{5} e^{2t} + \frac{18}{5} e^{3t} - 4\left(\frac{23}{5} e^{2t} + \frac{2}{5} e^{3t}\right) = 2e^{3t}$

$\cancel{\frac{92}{5} e^{2t}} + \frac{18}{5} e^{3t} - \cancel{\frac{92}{5} e^{2t}} - \frac{8}{5} e^{3t} = 2e^{3t}$

works $\frac{10}{5} e^{3t} = 2e^{3t}$

Outstanding!