Exam 1 Calc 1 2/10/2006

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the definition of the derivative of the function f(x) at the point $x = a$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

Good

Use the graph of f(x) at the bottom of the page for problems 2 and 3:

a) What is f(7)? 1

b) What is $\lim_{x\to 7^+} f(x)$? -1

Excellent

c) What is $\lim_{x\to 7} f(x)$? $\lim_{x\to 7^+} (f(x)) = -1$ $\lim_{x\to 7^-} f(x) = 1$ $\boxed{\lim_{x\to 7} f(x) = \text{does not exist}}$

3. a) For which value(s) of x is f(x) not continuous? Why?

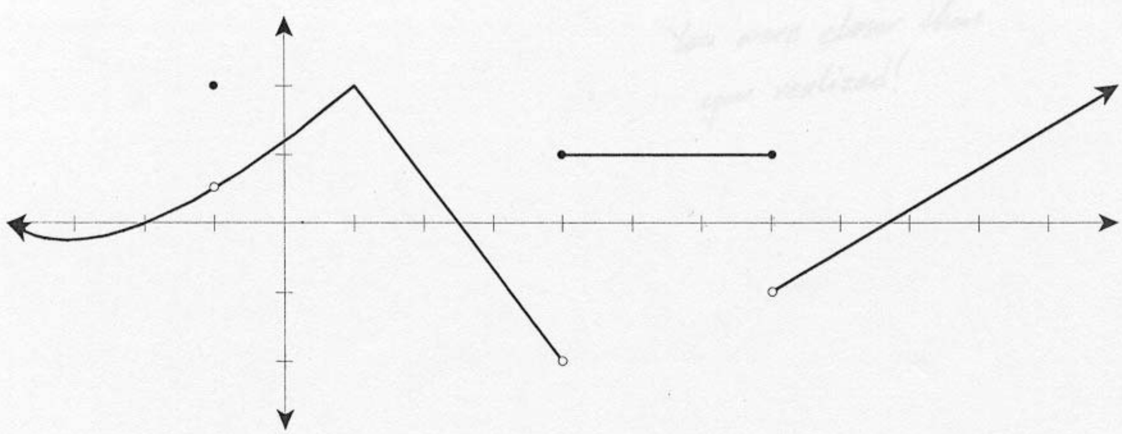
(-1, 4, 7) There are open circles therefore the graph is showing a jump in values and is not continuous

b) For which value(s) of x is f(x) not differentiable? Why?

Excellent

1 · not differentiable a V corners

(-1, 4, 7) - if its not continuous then its also not differentiable



4. Let $f(x) = 2^x$. Estimate the value of $f'(0)$ by using the definition of the derivative and taking successively smaller values of h.

$$f'(a) = \lim \frac{f(a+h) - f(a)}{h}$$

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{2^h - 2^0}{h}$$

$$= \lim_{h \to 0} \frac{2^h - 1}{h}$$

$$\boxed{= 0.693147}$$

Great

let $g(h) = \dfrac{2^h - 1}{h}$

h	g(h)
.1	1
0.1	0.717734
0.01	0.695555
0.001	0.693387
0.0001	0.693171 2
0.00001	0.6931496
0.000001	0. 693147

It getting closer and closer to 0.693147

5. Evaluate $\lim_{x \to -7} \dfrac{\frac{1}{7} + \frac{1}{x}}{7 + x}$ exactly.

$$\lim_{x \to -7} \frac{\frac{1}{7}\frac{x}{x} + \frac{1}{x} \cdot \frac{7}{7}}{7 + x} = \lim_{x \to -7} \frac{\cancel{x+7}}{7x} \cdot \frac{1}{\cancel{x+7}} = \lim_{x \to -7} \frac{1}{7x} \quad \frac{1}{7 \cdot -7} = -\frac{1}{49}$$

$$\lim_{x \to -7} \frac{\frac{1}{7} + \frac{1}{x}}{7 + x} = \boxed{-\frac{1}{49}}$$

Well done

6. If f(x) = $\sqrt{x+1}$ Use the definition of the derivative to find f′(2).

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{2+h+1} - \sqrt{2+1}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{3+h} - \sqrt{3}}{h} \cdot \frac{(\sqrt{3+h} + \sqrt{3})}{\sqrt{3+h} + \sqrt{3}}$$

$$= \lim_{h \to 0} \frac{3+h-3}{h(\sqrt{3+h} + \sqrt{3})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{3+h} + \sqrt{3})}$$

$$\lim_{h \to 0} \frac{1}{\sqrt{3+h} + \sqrt{3}}$$

$$= \frac{1}{\sqrt{3+0} + \sqrt{3}}$$

$$= \frac{1}{\sqrt{3} + \sqrt{3}}$$

$$= \frac{1}{2\sqrt{3}}$$

Nice Job!

7. Evaluate the limit $\lim_{x\to 1}\frac{4-3x}{x^2+2}$ and justify each step by indicating the appropriate limit law(s) from the list below.

$$\lim_{x\to 1}\frac{4-3x}{x^2+2} \xrightarrow{\text{Quotient rule}} \frac{\lim_{x\to 1} 4-3x}{\lim_{x\to 1} x^2+2} \xrightarrow{\text{difference rule}} \frac{\lim_{x\to 1} 4 - \lim_{x\to 1} 3x}{\lim_{x\to 1} x^2+2}$$

Sum rule

$$\frac{\left(\lim_{x\to 1} 4\right)-\left(\lim_{x\to 1} 3x\right)}{\left(\lim_{x\to 1} x^2\right)+\left(\lim_{x\to 1} 2\right)} \xrightarrow{\text{Constant multiple rule}} \frac{\left(\lim_{x\to 1} 4\right)-\left(3\lim_{x\to 1} x\right)}{\left(\lim_{x\to 1} x^2\right)+\left(\lim_{x\to 1} 2\right)} \xrightarrow{\text{power rule}} \frac{\left(\lim_{x\to 1} 4\right)-\left(3\lim_{x\to 1} x\right)}{\left(\lim_{x\to 1} x\right)^2+\left(\lim_{x\to 1} 2\right)}$$

Constant Rule

$$\frac{(4)-\left(3\lim_{x\to 1} x\right)}{\left(\lim_{x\to 1} x\right)^2+2} \xrightarrow{\text{rule X}} \frac{(4)-3(1)}{(1)^2+2} \xrightarrow{\text{Basic Math}} \frac{(4-3)}{(1+2)} = \frac{1}{3}$$

Nice.

Algebraic Limit Properties

Let c be a constant. Then as long as $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist,

Constant Rule for Limits: $\lim_{x\to a} c = c$

Rule X for Limits: $\lim_{x\to a} x = a$

Sum Rule for Limits: $\lim_{x\to a}\left[f(x)+g(x)\right] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

Difference Rule for Limits: $\lim_{x\to a}\left[f(x)-g(x)\right] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

Constant Multiple Rule for Limits: $\lim_{x\to a}\left[c\cdot f(x)\right] = c\cdot\lim_{x\to a} f(x)$

Product Rule for Limits: $\lim_{x\to a}\left[f(x)\cdot g(x)\right] = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$

Quotient Rule for Limits: $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ as long as $\lim_{x\to a} g(x) \neq 0$.

Power Rule for Limits: $\lim_{x\to a}\left[f(x)\right]^n = \left[\lim_{x\to a} f(x)\right]^n$

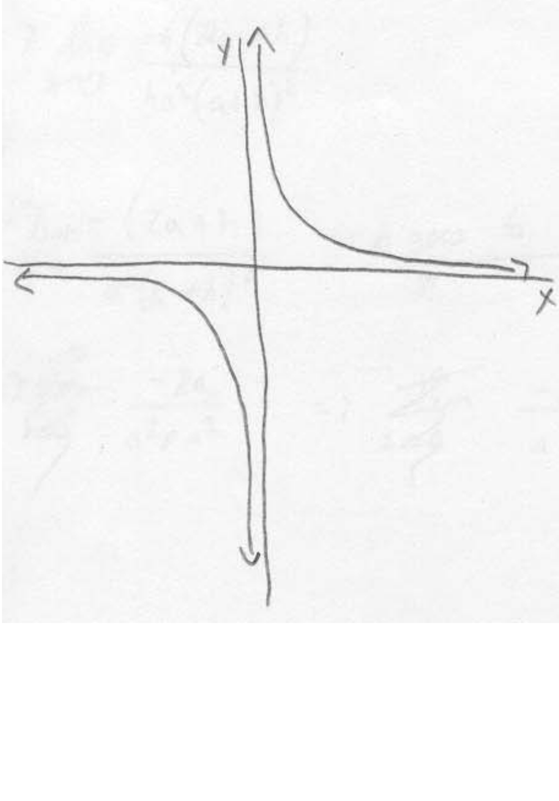
8. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod, we had this test, and it was so messed up. Everything's multiple guess, of course, because there's like a million people in the class, but that means if you get something wrong you get no credit at all, and they won't tell you how to do it, they just put up the right answers afterwards so you learn *nothing*. So there was this question about, like, estimating the slope of the tangie-thing for $1/x$ where $x = -1$, right? So I did it like I remembered from class, where I did the slope from (-1,-1) to the point where x is zero, 'cause it's easy to plug in zero, right? So it came out undefined, so I marked that answer, but they said it was wrong. I have no idea why!"

Explain to Bunny, as clearly as possible, either what issues keep her answer from being correct, or why her answer is in fact right so she can go argue with the professor.

Bunny needs to use a value less than zero to estimate the slope of $f(x) = 1/x$ at $(-1,-1)$. This is because x cannot equal zero using the given function (cannot divide by zero). This is why a vertical asymptote is formed, approaching $-\infty$ as $x \to 0^-$ (as shown in the graph).

Bunny should use values much closer to $x=-1$, such as $x = -9/10$ or $x = -11/10$. This way her secant line will be a much closer estimate of the line tangent to $x=-1$.

Wonderful.



9. Let $f(x) = \frac{1}{x^2}$. Use the definition of the derivative to find $f'(a)$.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{(a+h)^2} - \frac{1}{a^2}}{h}$$

$$= \lim_{h \to 0} \frac{a^2 - (a+h)^2}{(a+h)^2 a^2} \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{a^2} - \cancel{a^2} - 2ah - 2h^2}{(a+h)^2 a^2 \cdot h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-2a - 2h)}{(a+h)^2 a^2 \cdot \cancel{h}}$$

$$= \frac{-2a}{a^4}$$

$$= -\frac{2}{a^3}$$

10. Evaluate $\lim_{x\to\infty}\left(\sqrt{x^2+ax+6}-\sqrt{x^2+3x+1}\right)$, where a is some constant.

$$\lim_{x\to\infty} \sqrt{x^2+ax+6} - \sqrt{x^2+3x+1}$$

$$= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+ax+6}-\sqrt{x^2+3x+1}\right)\left(\sqrt{x^2+ax+6}+\sqrt{a^2+3x+1}\right)}{\sqrt{x^2+ax+6}+\sqrt{x^2+3x+1}}$$

$$\lim_{x\to\infty} \frac{\cancel{x^2}+ax+6-\cancel{x^2}-3x-1}{\sqrt{x^2+ax+6}+\sqrt{x^2+3x+1}}$$

$$\lim_{x\to\infty} \frac{ax-3x+5}{\sqrt{x^2+ax+6}+\sqrt{x^2+3x+1}}$$

$$\lim_{x\to\infty} \frac{a-3+5/x}{\sqrt{1+a/x+6/x^2}+\sqrt{1+3/x+1/x^2}}$$

$$\lim_{x\to\infty} \frac{a-3-0}{\sqrt{1+0+0}+\sqrt{1+0+0}}$$

$$= \frac{a-3}{\sqrt{1}+\sqrt{1}}$$

$$= \boxed{\frac{a-3}{2}}$$

Nice Job