Exam 4 Practice   Calc 1   2/20/2006

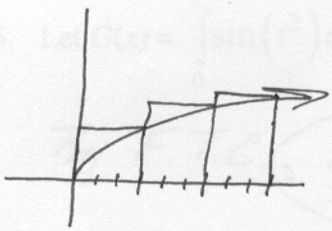
Each problem is worth 0 points. For full credit provide complete justification for your answers.

1. Suppose that the following table of values gives a car's speed as it accelerates after a light turns green, then brakes to a stop at the next intersection. Based on this data, give an upper approximation for the distance the car traveled between the two intersections.

t (seconds)	0	5	10	15
speed (feet/second)	5	40	50	10

Upper Sum $= 40\cdot 5 + 50\cdot 5 + 50\cdot 5 = 700$ feet

2. Use a right-hand sum with three equal subintervals to approximate the value of $\int_0^9 \sqrt{x}\,dx$.



$$R_3 = f(3)\cdot 3 + f(6)\cdot 3 + f(9)\cdot 3$$
$$= \sqrt{3}\cdot 3 + \sqrt{6}\cdot 3 + \sqrt{9}\cdot 3$$
$$= 3\sqrt{3} + 3\sqrt{6} + 9$$
$$\approx 21.5$$

3. Evaluate $\int_0^9 \sqrt{x}\,dx$ exactly.

$$\int_0^9 x^{1/2}\,dx = \frac{2}{3}x^{3/2}\Big|_0^9 = \frac{2}{3}\cdot 9^{3/2} - \frac{2}{3}\cdot 0^{3/2}$$
$$= \frac{2}{3}\cdot 27$$
$$= 18$$

4. Let $F(x) = \int_0^x \cos\theta \, d\theta$.

a. Evaluate $F(\pi/2)$.

$$F\left(\frac{\pi}{2}\right) = \int_0^{\pi/2} \cos\theta \, d\theta = \sin\theta \Big|_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0$$

$$= 1$$

b. Evaluate $F(2\pi)$.

$$F(2\pi) = \int_0^{2\pi} \cos\theta \, d\theta = \sin\theta \Big|_0^{2\pi} = \sin 2\pi - \sin 0$$

$$= 0 - 0$$

$$= 0$$

5. Let $G(x) = \int_0^x \sin\left(t^2\right) dt$. What is $G'(x)$?

By F.T.C., $G'(x) = \sin(x^2)$.

6. Find $\displaystyle\int \frac{x}{1+x^2}\,dx = \int \frac{\cancel{x}}{u}\cdot\frac{du}{2\cancel{x}}$

let $u = 1+x^2$

$\frac{du}{dx} = 2x$

$\frac{du}{2x} = dx$

$= \frac{1}{2}\int \frac{1}{u}\,du$

$= \frac{1}{2}\cdot \ln|u| + C$

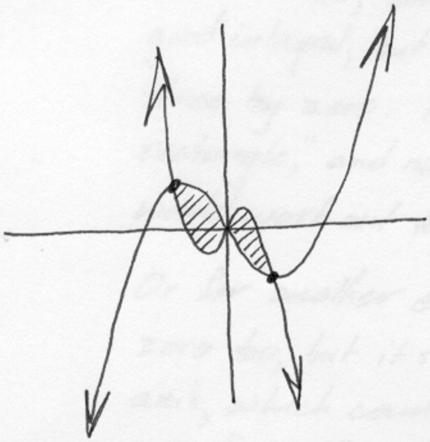
$= \frac{1}{2}\ln|1+x^2| + C$

Check: $\left(\frac{1}{2}\ln|1+x^2| + C\right)'$

$= \frac{1}{2}\cdot\frac{1}{1+x^2}\cdot 2x + 0$

$= \frac{x}{1+x^2}$ ✓

7. Find the area bounded between $y = x^3 - 8x$ and $y = -x^3$.



Where do they cross?

$x^3 - 8x = -x^3$

$2x^3 - 8x = 0$

$2x(x^2 - 4) = 0$

$2x(x+2)(x-2) = 0$

$x = 0,\ x = -2,\ x = 2$

$$\text{So Area} = \int_{-2}^{0}\left[(x^3-8x) - (-x^3)\right]dx + \int_{0}^{2}\left[(-x^3) - (x^3-8x)\right]dx$$

$$= \int_{-2}^{0}(2x^3 - 8x)\,dx + \int_{0}^{2}(8x - 2x^3)\,dx$$

$$= \left[\frac{x^4}{2} - 4x^2\right]_{-2}^{0} + \left[4x^2 - \frac{x^4}{2}\right]_{0}^{2}$$

$$= (0) - (8-16) + (16-8) - (0)$$

$$= 16$$

8. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Dude, I love these multiple choice tests! We had this one in calculus, and I was pretty much just guessing on some of the stuff, but it's so cool because you can totally eliminate some of the answers even if you've got no clue how to do the problem. Like this one, there was one of them integrals, and one of the answers was zero. So like normally they give the area, right, except they give the negative area if it's down below or whatever, but there's no way an area can be zero, right? So I just guessed from all the answers that aren't zero, and my odds are way better, right?"

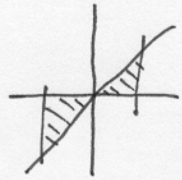
Explain clearly to Biff whether this reasoning is valid or not.

(There are always lots of very different good ways to respond to these questions, so taking another approach isn't necessarily bad, but I'd probably go with something like...)

Well Biff, you've got <u>some</u> pieces right, but overall you need some serious help. Of course an actual area shouldn't normally be zero, but definite integrals are a bit more involved than areas.

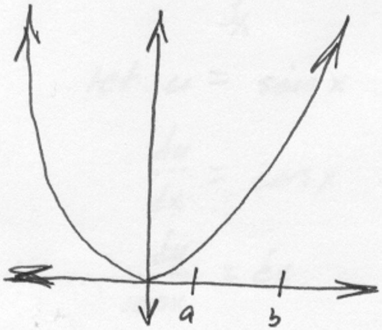
First of all, think about something like $\int_2^2 3\,dx$. It's a perfectly good integral, but the region it represents would be a rectangle that's three by zero. Now you're probably saying "But that's not really a rectangle," and maybe you're right, but that's the point. That integral would work out to zero, and it only kind of corresponds to an area.

Or for another example, think about $\int_{-1}^{1} x\,dx$. That works out to zero too, but it's because there's just as much area above the axis, which counts positive, as below the axis, which counts negative. For the integral those cancel out, just like if you went forwards and then backwards, or if a business ran a profit and then a loss. But as areas, none of that cancellation makes sense at all – instead the actual areas would add up.



So the key, Biff, is that definite integrals aren't <u>just</u> areas, so normal things you expect from areas, like not being zero, don't carry over simply to integrals.

9. Write an expression using sigma notation to approximate the area under $f(x) = x^2$ but above the x-axis between $x = a$ and $x = b$ using n subdivisions.



The total distance from a to b is $b-a$

So broken into n parts, $\Delta x = \frac{b-a}{n}$

Then the first subinterval goes from a to $a + \frac{b-a}{n}$,

the second goes from $a + \frac{b-a}{n}$ to $a + \frac{b-a}{n} \cdot 2$,

and in general the i^{th} subinterval goes from

$a + \frac{b-a}{n}(i-1)$ to $a + \frac{b-a}{n} \cdot i$

So I'll use $x_i^* = a + \frac{b-a}{n} \cdot i$ as my representative point for the i^{th} subinterval.

Then the area of the i^{th} box is $f\left(a + \frac{b-a}{n} i\right) \cdot \frac{b-a}{n}$,

So adding up n of them would be

$$\sum_{i=1}^{n} f\left(a + \frac{b-a}{n} i\right) \cdot \frac{b-a}{n}$$

10. Show that $\int_{-\pi/6}^{\pi/6} \sin x \cos^n x \, dx$ is always zero for any positive integer n.

let $u = \cos x$

$\frac{du}{dx} = -\sin x$

$\frac{du}{-\sin x} = dx$

So $\int_{-\pi/6}^{\pi/6} \sin x (\cos x)^n dx = \int_{x=-\frac{\pi}{6}}^{x=\frac{\pi}{6}} \sin x \cdot u^n \cdot \frac{du}{-\sin x}$

$= -\int_{x=-\frac{\pi}{6}}^{x=\frac{\pi}{6}} u^n \, du$

$= -\left.\frac{u^{n+1}}{n+1}\right|_{x=-\pi/6}^{x=\pi/6}$

$= -\left.\frac{\cos^{n+1} x}{n+1}\right|_{-\pi/6}^{\pi/6}$

$= -\frac{1}{n+1} \cos^{n+1}\left(\frac{\pi}{6}\right) - -\frac{1}{n+1} \cos^{n+1}\left(\frac{-\pi}{6}\right)$

$= -\frac{1}{n+1} \cos^{n+1}\left(\frac{\pi}{6}\right) + \frac{1}{n+1} \cos^{n+1}\left(\frac{\pi}{6}\right)$

$= 0$

Because $\cos x$ is an even function, it's the same doing $\cos\left(\frac{\pi}{6}\right)$ as $\cos\left(\frac{-\pi}{6}\right)$.