Exam 2 In-class Portion   Differential Equations   3/17/06

Each problem is worth 10 points. For full credit indicate clearly how you reached your answer.

1. Determine whether $x(t) = -e^{-2t}\sin 3t$, $y(t) = e^{-2t}\cos 3t$ is a solution to the system of differential equations

$$\frac{dx}{dt} = -2x - 3y$$
$$\frac{dy}{dt} = \ \ 3x - 2y$$

10

product Rule

$x'(t) = 2e^{-2t} \cdot \sin 3t + -3e^{-2t}\cos 3t \overset{?}{=} -2(-e^{-2t}\sin 3t) - 3(e^{-2t}\cos 3t)$

Yes it is a sol for dy/dt. ✓

Great

$y'(t) = -2e^{-2t}\cos 3t + e^{-2t} \cdot (-3\sin 3t)$

$\overset{?}{=} 3(-e^{-2t}\sin 3t) - 2(e^{-2t}\cos 3t)$ Yes it is a sol for dy/dt ✓

2. Find all equilibrium points of the predator-prey system

$$\frac{dR}{dt} = 2R\left(1 - \frac{R}{2.5}\right) - 1.5RF$$
$$\frac{dF}{dt} = -F + 0.8RF$$

we have

$\begin{cases} R\left(2 - \frac{2R}{2.5} - 1.5F\right) = 0 \\ F(0.8R - 1) = 0 \end{cases} \Rightarrow$ $R = 0$ or $2 - \frac{2R}{2.5} - 1.5F = 0$; $F = 0$ or $0.8R - 1 = 0$

$\therefore$ when $R = 0$, $F = 0$.

Well done

$R = 2.5$, $F = 0 \Leftarrow 2 - \frac{2R}{2.5} = 0$

$R = \frac{5}{4}$, $F = \frac{2}{3} \Leftarrow \begin{cases} 2 - \frac{2R}{2.5} - 1.5F = 0 \\ 0.8R - 1 = 0 \end{cases}$

$\therefore$ $(0, 0)$, $(2.5, 0)$

$\left(\frac{5}{4}, \frac{2}{3}\right)$ are equilibrium points.

3. Find a general solution to the differential equation $y'' - y' - 12y = 0$.

$y'' - y' - 12y = 0$. This equation probably has a solution of the type $y(t) = e^{st}$

so, $y'(t) = se^{st}$

and $y''(t) = s^2 e^{st}$

$s^2 e^{st} - se^{st} - 12e^{st} = 0$

$e^{st}(s^2 - s - 12) = 0$ $\quad -1$

-12

$e^{st}(s-4)(s+3) = 0$ $\quad -4, 3$

so, general solution would be:

$$\boxed{y(t) = Ae^{4t} + Be^{-3t}}$$

Excellent

check:

$y(t) = Ae^{4t} + Be^{-3t}$

$y'(t) = 4Ae^{4t} + -3Be^{-3t}$

$y''(t) = 16Ae^{4t} + 9Be^{-3t}$

$16Ae^{4t} + 9Be^{-3t} - (4Ae^{4t} + (-3Be^{-3t})) - 12(Ae^{4t} + Be^{-3t}) \stackrel{?}{=} 0$.

$12Ae^{4t} + 12Be^{-3t} - 12Ae^{4t} - 12Be^{-3t} \stackrel{?}{=} 0$.

Yep, it works.

4. a) Find a general solution to the partially decoupled system

$$\frac{dx}{dt} = 3x + 2y$$

$$\frac{dy}{dt} = -2y$$

b) Find a particular solution satisfying the initial condition $(x_0, y_0) = (5,3)$.

A) $\frac{1}{y}dy = -2dt \Rightarrow \ln y = -2t + c \Rightarrow y = e^{-2t}e^{c} \Rightarrow$ Let $A = e^{c}$

So $\boxed{y = Ae^{-2t}}$

then plug into $\frac{dx}{dt}$ equation:

$\Rightarrow \frac{dx}{dt} = 3x + 2Ae^{-2t} \Rightarrow \left[\frac{dx}{dt} - 3x = 2Ae^{-2t}\right]$ now find integrating factor

$\Rightarrow \mu(t) = e^{\int -3dt} \Rightarrow \left[\mu(t) = e^{-3t}\right] \Rightarrow$ multiply through;

$\Rightarrow \frac{dx}{dt}e^{-3t} - 3xe^{-3t} = 2Ae^{-5t} \Rightarrow$ But $\frac{dx}{dt}e^{-3t} - 3xe^{-3t} = \frac{d}{dx}\left[xe^{-3t}\right]$

So integrate both sides with respect to t;

$xe^{-3t} = \int 2Ae^{-5t}\,dt \Rightarrow xe^{-3t} = -\frac{2A}{5}e^{-5t} + c$

$\Rightarrow \boxed{x = -\frac{2A}{5}e^{-2t} + ce^{3t} \quad y = Ae^{-2t}}$ general solution

Well done!

B) $(x_0, y_0) = (5,3)$

So at $t=0$; $5 = -\frac{2A}{5}e^{-2(0)} + ce^{3(0)} \Rightarrow \left[5 = -\frac{2A}{5} + c\right]$

And $3 = Ae^{-2(0)} = A$ $\qquad 5 = -\frac{2}{5}(3) + c \Rightarrow 5 + \frac{6}{5} = c \Rightarrow \boxed{c = 6.2}$

$\Rightarrow \boxed{A = 3}$ so

So $\boxed{x = -\frac{6}{5}e^{-2t} + 6.2e^{3t} \quad y = 3e^{-2t}}$ particular solution satisfying $(x_0, y_0) = (5,3)$

5. How do you know that Laplace transforms are linear, i.e. that $\mathscr{L}[a \cdot f(x) + b \cdot g(x)] = a\,\mathscr{L}[f(x)] + b\,\mathscr{L}[g(x)]$ for any functions f and g whose Laplace transforms exist?

$$\mathscr{L}\left[a f(x) + b \cdot g(x)\right] = \int_0^\infty \left[a f(x) + b g(x)\right] e^{-st}\, dt$$

$$= \int_0^\infty a f(x) \cdot e^{-st} + b \cdot g(x) e^{-st}\, dt$$

$$= \int_0^\infty a f(x) \cdot e^{-st}\, dt + \int b\, g(x) e^{-st}\, dt$$

$$= a \int_0^\infty f(x) e^{-st}\, dt + b \int g(x) e^{-st}\, dt$$

Now, $\mathscr{L}\left[f(x)\right] = \int_0^\infty f(x) \cdot e^{-st}\, dt$

and $\mathscr{L}\left[g(x)\right] = \int_0^\infty g(x) \cdot e^{-st}\, dt$

Wonderful

$\therefore$ Putting in these values

$$\mathscr{L}\left[a(f(x)) + b(g(x))\right] = a\,\mathscr{L}[f(x)] + b\,\mathscr{L}[g(x)]$$

Hence Laplace transforms are linear.

Proved