Exam 3 Differential Equations 4/14/06

Each problem is worth 10 points. For full credit indicate clearly how you reached your answer.

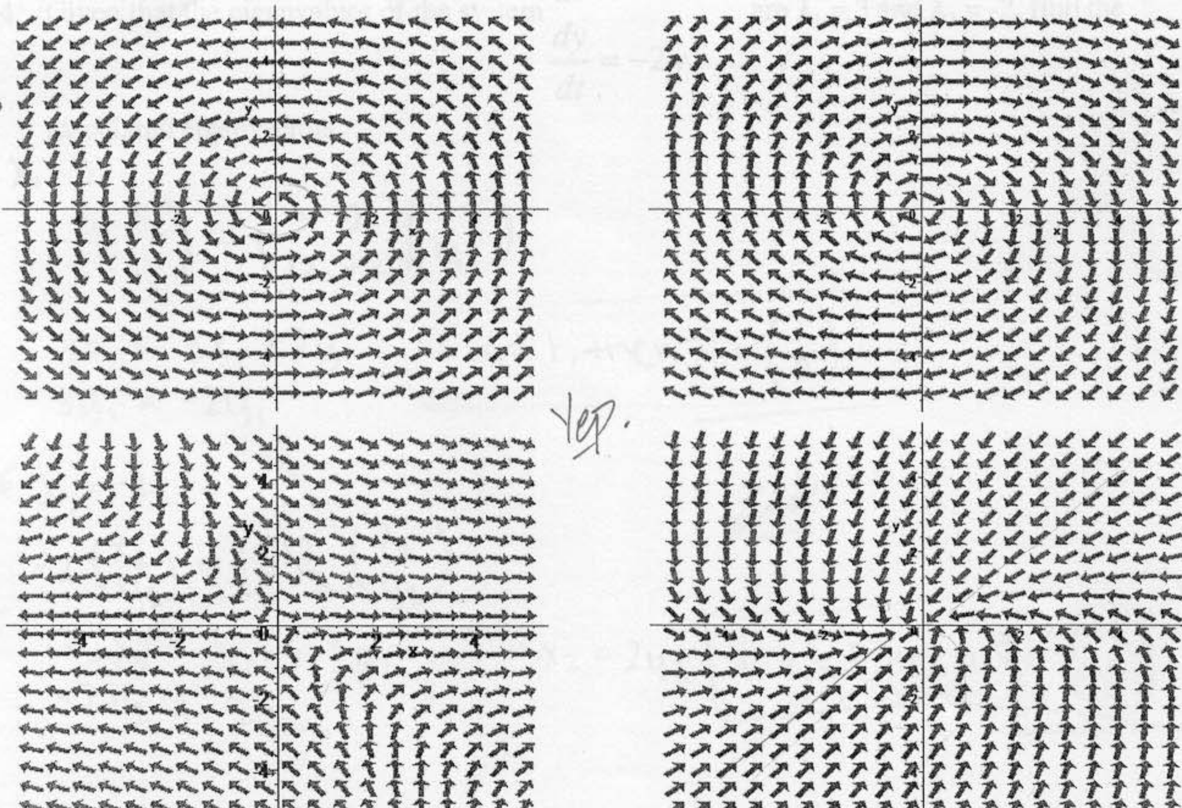
1. If a planar system of differential equations has eigenvalues $\lambda_1 = 3$, $\lambda_2 = -2$ and associated eigenvectors $\mathbf{v}_1 = (1,0)$ and $\mathbf{v}_2 = (3,-1)$, write a general solution to the system.

$$\hat{Y} = Ae^{3t}\begin{pmatrix}1\\0\end{pmatrix} + Be^{-2t}\begin{pmatrix}3\\-1\end{pmatrix}$$

Good.

0

2. Classify each of the following planar systems' equilibria as sources, sinks, saddles, centers, spiral sources, or spiral sinks.



center

spiral source

Yep.

saddle

sink

3. Find all eigenvalues of the system of differential equations $\begin{aligned}\frac{dx}{dt} &= -2x-3y \\ \frac{dy}{dt} &= 3x-2y\end{aligned}$.

$A = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix}$ $\quad \det(A - I\lambda) = \det\begin{pmatrix} -2-\lambda & -3 \\ 3 & -2-\lambda \end{pmatrix}$

$= (\lambda+2)^2 + 9 = (\lambda+2)^2 - (3i)^2$

$= (\lambda+2-3i)(\lambda+2+3i) = 0$

$\Rightarrow \lambda_1 = \underline{-2+3i}$

$\lambda_2 = \underline{-2-3i}$

Good.

4. Given that the eigenvalues of the system $\begin{aligned}\frac{dx}{dt} &= 3x+2y \\ \frac{dy}{dt} &= -2y\end{aligned}$ are $\lambda_1 = 3$ and $\lambda_2 = -2$, find the associated eigenvectors.

$\lambda_1 = 3$

$3x_1 + 2y_1 = 3x_1$

$-2y_1 = 3y_1$

$V_1 = \underline{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$ ← specific

$V_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ ← general

$\lambda_2 = -2$

$3x_2 + 2y_2 = -2x_2$

$-2y_2 = -2y_2, \; y_2 = 1$

$5x_2 = -2y_2$

$x_2 = -\frac{2}{5}y_2$

general → $V_2 = \begin{pmatrix} -2y_2 \\ 5y_2 \end{pmatrix}$

Great!

specific → $V_2 = \underline{\begin{pmatrix} -2 \\ 5 \end{pmatrix}}$

5. Express the eigenvalues of a planar system with coefficient matrix $A = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}$.

$$\lambda \hat{Y} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \hat{Y}$$

$$0 = \hat{Y}\left[\begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right]$$

$$0 = \hat{Y}\begin{pmatrix} -\lambda & 1 \\ -q & -p-\lambda \end{pmatrix}$$

$$0 = (-\lambda)(-p-\lambda) - (1)(-q)$$

$$0 = p\lambda + \lambda^2 + q$$

$$0 = \lambda^2 + p\lambda + q$$

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4(1)q}}{2}$$

eigenvalues $\boxed{\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}}$

Yes

6. Given that a planar system with coefficient matrix $\begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix}$ has

$$\mathbf{Y} = e^t \begin{pmatrix} \cos t - \sin t \\ 2\cos t \end{pmatrix} + i e^t \begin{pmatrix} \cos t + \sin t \\ 2\sin t \end{pmatrix}$$

As a solution, find a particular solution meeting the initial condition $\mathbf{Y}_0 = (2,0)$.

$\hat{A} = \begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix}$

so, $\hat{y}(t) = k_1 e^t \begin{pmatrix} \cos t - \sin t \\ 2\cos t \end{pmatrix} + k_2 e^t \begin{pmatrix} \cos t + \sin t \\ 2\sin t \end{pmatrix}$

plug in $t=0$ because of initial condition

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} = k_1 e^0 \begin{pmatrix} \cos 0 - \sin 0 \\ 2\cos 0 \end{pmatrix} + k_2 e^0 \begin{pmatrix} \cos 0 + \sin 0 \\ 2\sin 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Nice!

$2 = k_1 + k_2$

$0 = 2k_1 \quad \Rightarrow \quad k_1 = 0$, so $k_2 = 2$

since $k_1 = 0$, the first half of the equation disappears

$$\boxed{\hat{y}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2e^t \begin{pmatrix} \cos t + \sin t \\ 2\sin t \end{pmatrix}, \square}$$

7. Prove that if λ is an eigenvalue of a matrix $\mathbf{A}$, with corresponding eigenvector $\mathbf{v}$, then $\mathbf{Y} = e^{\lambda t}\mathbf{v}$ is a solution to $\dfrac{d\mathbf{Y}}{dt} = \mathbf{AY}$.

Pf: $\because \vec{Y} = e^{\lambda t}\vec{v}$

$$\therefore \frac{d\vec{Y}}{dt} = \underline{\lambda \cdot e^{\lambda t} \cdot \vec{v}}$$

Plug into $\frac{d\vec{Y}}{dt} = \vec{A}\vec{Y}$, then

$$(\lambda \cdot e^{\lambda t} \cdot \vec{v}) \stackrel{?}{=} \vec{A} \cdot \vec{Y}$$

$$\Rightarrow \vec{A}\vec{Y} = \underline{\vec{A} \cdot e^{\lambda t}\vec{v}}$$

$$= e^{\lambda t} \cdot \vec{A} \cdot \vec{v}$$

$$= e^{\lambda t} \cdot \underline{\lambda \vec{V}}$$

$$= \lambda \cdot e^{\lambda t} \cdot \vec{v}$$

By definition of eigenvalues.

Equal.

Hence, it works.

Good

8. Find an eigenvalue and the corresponding eigenvector for the matrix $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}$.

$(0-\lambda)(0-\lambda) - (-1)(3)$

$\lambda^2 + 3 = 0$

$\lambda^2 = -3$

$\lambda = \pm\sqrt{-3}$

$\boxed{\lambda = \pm i\sqrt{3}}$ $\lambda_1 = i\sqrt{3}$ and $\lambda_2 = -i\sqrt{3}$

using $\lambda_1 = i\sqrt{3}$

$$i\sqrt{3}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$-i\sqrt{3}x_1 = y_1 \Leftarrow i\sqrt{3}x_1 = -y_1$

$i\sqrt{3}y_1 = 3x_1$

$\boxed{\text{if } x_1 = 1,\ \vec{v} = \begin{pmatrix} 1 \\ -i\sqrt{3} \end{pmatrix}}$

$i\sqrt{3}(-i\sqrt{3}) \stackrel{?}{=} 3$ check

$1 \cdot 3 = 3$ yes correct

Nice!

9. For what values of a and b will the planar system $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} a & b \\ b & 0 \end{pmatrix} \mathbf{Y}$ have a center?

$$(a-\lambda)(0-\lambda) - (b)(b) = 0$$

$$-a\lambda + \lambda^2 - b^2 = 0$$

$$\lambda^2 - a\lambda - b^2 = 0$$

$$\lambda = \frac{-(-a) \pm \sqrt{(-a)^2 - 4(1)(-b^2)}}{2(1)}$$

$$= \frac{a \pm \sqrt{a^2 + 4b^2}}{2}$$

Now a center corresponds to purely imaginary eigenvalues, so the "a" above would need to be zero and $a^2 + 4b^2$ would need to be negative. But squares (of reals) can't be negative, so there are no values of a and b that make this system have a center!