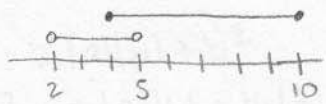
Examlet 2 Foundations of Advanced Math 2/24/06

1. a) Find $\{1, 2, 3, 4\} \cap \{3, 4, 5, 6\}$.

$\{3, 4\}$

b) Find $(2, 5) \cup [4, 10]$.

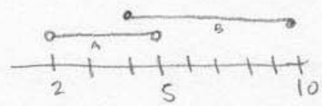


$(2, 10]$

c) Find $(2, 5) - [4, 10]$.

A B

$\{x \mid x \in A \wedge x \notin B\}$



$(2, 4)$

d) Find $\{3, 7\} \times \{-1, 0, 1\}$, where "×" indicates the Cartesian product.

$\{(3,-1), (3,0), (3,1), (7,-1), (7,0), (7,1)\}$

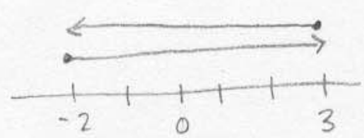
e) Solve the inequality $|2x - 1| \le 5$, and write your answer as an interval or union of intervals.

$|2x-1| \le 5$

$-5 \le 2x - 1 \le 5$ Add 1

$\frac{-4}{2} \le \frac{2x}{2} \le \frac{6}{2}$ divide by 2

$-2 \le x \le 3$



$[-2, 3]$ ■

Great

2. State and prove the triangle inequality.

State: $|x|+|y| \geqslant |x+y|$

we know $2|x||y| \geqslant 2xy$ (because $|x||y|$ always positive. but xy could be negative.)

$x^2 + 2|x||y| + y^2 \geqslant x^2 + 2xy + y^2$.

we know $|x|^2 = x^2$

$\Rightarrow |x|^2 + 2|x||y| + |y|^2 \geqslant x^2 + 2xy + y^2$

$(|x|+|y|)^2 \geqslant (x+y)^2$

If $a^2 \geqslant b^2$

$\Rightarrow \sqrt{a^2} \geqslant \sqrt{b^2} \Rightarrow |a| \geqslant |b|$.

$\Rightarrow \sqrt{(|x|+|y|)^2} \geqslant \sqrt{(x+y)^2}$

$\Rightarrow \big||x|+|y|\big| \geqslant |x+y|$.

but $|x|+|y| \geq 0 \Rightarrow |x|+|y| = \big||x|+|y|\big|$

$\Rightarrow |x|+|y| \geqslant |x+y|$.

$\Rightarrow$ it is proved.

Excellent

3. Let $\boxed{\{A_i \mid i \in I\}}$ be an indexed family of sets , and let B be any set, all subsets of some universal set U. Show that $B \cap \bigcup_{i \in I} A_i = \bigcup_{i \in I} (B \cap A_i)$.

$$B \cap \bigcup_{i \in I} A_i = \bigcup_{i \in I} (B \cap A_i)$$

Let $x \in B \cap \bigcup_{i \in I} A_i$. So $x \in B$ and, for some i_0, $x \in A_i$. So, for some i_0, $x \in \bigcup_{i \in I} (B \cap A_i)$. So,

$$\boxed{B \cap \bigcup_{i \in I} A_i \subseteq \bigcup_{i \in I} (B \cap A_i)}$$

Let $x \in \bigcup_{i \in I} (B \cap A_i)$. So for some i_0, $x \in B \cap A_i$. Thus $x \in B$ and, for some i_0, $x \in A_i$. Thus $x \in B \cap \bigcup_{i \in I} A_i$. So,

$$\boxed{\bigcup_{i \in I} (B \cap A_i) \subseteq B \cap \bigcup_{i \in I} A_i}$$

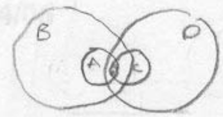
Well done!

$$B \cap \bigcup_{i \in I} A_i = \bigcup_{i \in I} (B \cap A_i)$$

4. If A and B are bounded sets of real numbers, $A \cup B$ is bounded as well.

Proof: Well, since A is bounded $\exists N_a \in \mathbb{R}$ such that $\forall a \in A$, $|a| \leq N_a$. Similarly since B is bounded $\exists N_b \in \mathbb{R}$ such that $\forall b \in B$, $|b| \leq N_b$. Now let N be the larger of N_a and N_b. Then for any $x \in A \cup B$, either $x \in A$, in which case $|x| \leq N_a \leq N$, or $x \in B$, in which case $|x| \leq N_b \leq N$, or both. In any case, $|x| \leq N$, so $A \cup B$ is bounded by N, as desired. $\square$

5. Let A, B, C, and D be sets. If $A \subseteq B$ and $C \subseteq D$, then $A \cap C \subseteq B \cap D$.



let $x \in A$, so for all x, $x \in B$

let $y \in C$, so for all y, $y \in D$

$n \in A \cap C$ $\quad$ $m \in B \cap D$

$n \in A \wedge n \in C$ $\quad$ $m \in B \wedge m \in D$

$n \in B \wedge n \in D$

$n \in B \cap D$

let $n \in A \cap C$

so $n \in A \wedge n \in C$

since $A \subseteq B$, $(\forall x \in A)(x \in B)$

and since $C \subseteq D$, $(\forall y \in C)(y \in D)$

so $n \in B \wedge n \in D$

$n \in B \cap D$ as desired.

Excellent