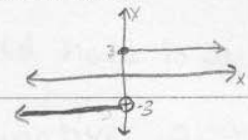
Examlet 3 Foundations of Advanced Math 3/24/06

Each problem is worth 10 points. Appropriate justification is required for full credit.

1. For the following, complete proofs that properties **do** hold are not necessary but justification when they **do not** hold is expected. Be explicit about the domains and codomains:

a) Give an example of a function which is neither one-to-one nor onto.

$f: \mathbb{R} \to \mathbb{R}$ $f(x) = \begin{cases} 3 & x \geq 0 \\ -3 & x < 0 \end{cases}$



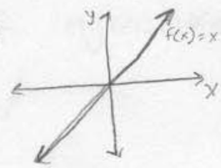
$f(5) = f(6)$ so the function is not one-to-one.

The only values hit in the codomain are -3 and 3. since the rest aren't the function is not onto.

Good.

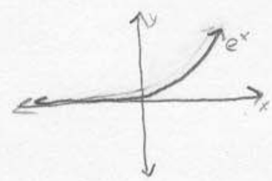
b) Give an example of a function which is both one-to-one and onto.

$f: \mathbb{R} \to \mathbb{R}$ $f(x) = x$



c) Give an example of a function which is one-to-one but not onto.

$f: \mathbb{R} \to \mathbb{R}$ $f(x) = e^x$



This function does not hit any of the negative numbers in the codomain so f is not onto.

Excellent

2. Prove that any function $f: \mathbb{R} \to \mathbb{R}$ of the form $f(x) = mx + b$, for constant real numbers m and b (with $m \neq 0$), is a surjection.

onto

Pick an arbitrary element of the codomain of f. Let's call it s.

Let r be an element of the domain where $r = \frac{s-b}{m}$.

since $m \neq 0$ this is ok.

$$f(x) = mx + b$$

$$f(r) = \cancel{m}\frac{s-b}{\cancel{m}} + b$$

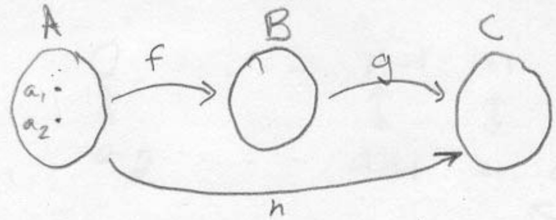
$$f(r) = s \cancel{- b + b}$$

$$f(r) = s$$

So for any arbitrary element of the codomain, s, there exists a pre-image that will result in that s. Therefore f is surjective.

Very well done.

3. Let $f: A \to B$, $g: B \to C$. Prove that if f and g are injective, then $h = g \circ f$ is injective.



let a_1 and a_2 be elements of the set A

suppose $h(a_1) = h(a_2)$

$g \circ f(a_1) = g \circ f(a_2)$

$g(f(a_1)) = g(f(a_2))$

$\therefore f(a_1) = f(a_2)$ since g is injective

$\therefore a_1 = a_2$ since f is injective.

$\therefore h$ must be injective because no two different arguements can have the same value.

Excellent

4. Suppose that A is a denumerable set, and $x \in A$. Show that $A - \{x\}$ is denumerable.

Well, since A is denumerable, there exists a bijection $f: \mathbb{N} \to A$. Now since f is onto and $x \in A$, there must be some $n_x \in \mathbb{N}$ for which $f(n_x) = x$. Then construct a new function as follows:

$$g(n) = \begin{cases} f(n) & \text{if } n < n_x \\ f(n+1) & \text{if } n \geq n_x \end{cases}$$

Then g is one-to-one because both branches are one-to-one and can't overlap, and onto since f was onto and g has all the same images except x. $\square$

The idea:

$$\begin{array}{ccccc}
\underline{A} & & \underline{\mathbb{N}} & & \underline{A - \{x\}} \\
f(1) & \longleftrightarrow & 1 & \longleftrightarrow & f(1) \\
f(2) & \longleftrightarrow & 2 & \longleftrightarrow & f(2) \\
f(3) & \longleftrightarrow & 3 & \longleftrightarrow & f(3) \\
\vdots & & \vdots & & \vdots \\
f(n_x-2) & \longleftrightarrow & n_x-2 & \longleftrightarrow & f(n_x-2) \\
f(n_x-1) & \longleftrightarrow & n_x-1 & \longleftrightarrow & f(n_x-1) \\
f(n_x) & \longleftrightarrow & n_x & \longleftrightarrow & f(n_x+1) \\
f(n_x+1) & \longleftrightarrow & n_x+1 & \longleftrightarrow & f(n_x+2) \\
f(n_x+2) & \longleftrightarrow & n_x+2 & \longleftrightarrow & f(n_x+3) \\
\vdots & & \vdots & & \vdots
\end{array}$$

5. Let $f: \mathbb{R} \to \mathbb{R}$ be a decreasing function.

a) What can be said about $f \circ f$?

Well, since f is decreasing, $x > y \Rightarrow f(x) < f(y)$.

So $x > y \Rightarrow f(x) < f(y) \Rightarrow f(y) > f(x) \Rightarrow f(f(y)) < f(f(x))$

$\Rightarrow f \circ f(x) > f \circ f(y)$

Thus $f \circ f$ is an increasing function.

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b) Suppose that f is composed with itself n times, and denote the resulting function $\overset{n}{\bigcirc} f(x)$.

What can be said about $\overset{n}{\bigcirc} f$?

Lemma 1: If f and g are increasing real functions, $f \circ g$ is increasing.

Proof: $x > y \Rightarrow g(x) > g(y) \Rightarrow f(g(x)) > f(g(y))$. □

Lemma 2: If f is increasing and g is decreasing, $f \circ g$ is decreasing.

Proof: $x > y \Rightarrow g(x) < g(y) \Rightarrow g(y) > g(x) \Rightarrow f(g(y)) > f(g(x))$. □

Proposition: $\overset{n}{\bigcirc} f(x)$ is increasing if n is even and decreasing if n is odd.

Proof: Well, let's induct. $\overset{1}{\bigcirc} f$ is decreasing by hypothesis, and $\overset{2}{\bigcirc} f$ is increasing by part a.

Then s'pose it's true for $\overset{k}{\bigcirc} f$. Then $\overset{k+1}{\bigcirc} f$ is decreasing when k is even by Lemma 2, and $\overset{k+1}{\bigcirc} f$ is increasing when k is odd by Lemma 1, as desired. □