Examlet 1b Foundations of Advanced Math 2/2/07

. a) State the definition of an even integer.

AN EVEN INTEGER IS ONE SUCH THAT IT CAN BE EXPRESSED IN THE FORM 2n, WHERE n IS AN INTEGER.

Good.

b) Is the statement "$(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x - y = 3)$" true or false? Support your answer.

FOR ALL REAL X, THERE EXISTS SOME REAL Y SUCH THAT $x-y=3$.

THIS IS TRUE. CHOOSE ANY REAL X. THEN, FIND THE REAL Y WHICH IS OBTAINED BY SUBTRACTING 3 FROM X. THESE X AND Y SATISFY $x-y=3$. (A SUITABLE Y CAN BE FOUND FROM ANY X.)

Excellent

2. a) Make a truth table for the statement P $\vee$ Q.

P	Q	P $\vee$ Q
T	T	T
T	F	T
F	T	T
F	F	F

Good

b) Determine whether the propositional $(P \wedge Q) \Rightarrow R$ is equivalent to $(P \Rightarrow R) \wedge (Q \Rightarrow R)$.

P	Q	R	$P \wedge Q$	$(P \wedge Q) \Rightarrow R$	$P \Rightarrow R$	$Q \Rightarrow R$	$(P \Rightarrow R) \wedge (Q \Rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	T	F	T	F
F	T	T	F	T	T	T	T
F	T	F	F	T	T	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T
				*			*

Using truth tables we see that the columns under $(P \wedge Q) \Rightarrow R$ is not the same as the column under the column $(P \Rightarrow R) \wedge (Q \Rightarrow R)$. Therefore the two propositionals are not equivalent. To distribute the $\Rightarrow R$ to the P and Q you would have to switch any $\wedge$ (and) signs to $\vee$ (or) signs so $(P \wedge Q) \Rightarrow R$ is equivalent to $(P \Rightarrow R) \vee (Q \Rightarrow R)$, but that is not what it is asking.

Extra Great!

3. Show that if n is an integer for which n^3 is even, then n is even.

Proof. If n is an integer for which n^3 is even, then n must be even. If n were an odd integer of the form $n = 2m+1$, $m \in \mathbb{Z}$, then

$$n^3 = (2m+1)^3 = 2(4m^3 + 6m^2 + 3m) + 1$$

$4m^3 + 6m^2 + 3m$ is an integer by closure of integers lets call it p. $(2m+1)^3 = 2p+1$ which is an odd integer. So if n were odd, n^3 is odd which proves then that for n^3 to be even, n must be even. $\square$

Excellent

4. Show that if x is a non-zero rational and y is irrational, then $x \cdot y$ is irrational.

Assume xy is rational. Then it may be written:

$$xy = \frac{a}{b} \quad a, b \in \mathbb{Z} \wedge b \neq 0$$

Since x is rational by the definition let $x = \frac{q}{r}$ $q, r \in \mathbb{Z}$ and $r \neq 0$

$$\frac{q}{r} \cdot y = \frac{a}{b}$$

$$y = \frac{a \cdot r}{q \cdot b}$$ let $a.r = c$ and $q.b = d$

$$y = \frac{c}{d}$$ $c, d \in \mathbb{Z}$ by C.O.I. and $d \neq 0$

which means that y is rational. But, the definition of ~~x~~ y is that it is irrational so the assumption of $x \cdot y$ being rational is incorrect and thus, $x \cdot y$ is irrational.

Excellent.

5. Show that $(\forall n \in \mathbb{N})(5 \mid (6^n - 1))$.

Let's induct!

When $n=1$ the statement is $5 \mid (6^1 - 1)$, or $5 \mid 5$, which is true since $5 \cdot 1 = 5$.

Now s'pose it's true for $n=k$, so $5 \mid (6^k - 1)$, meaning $5p = 6^k - 1$ for some $p \in \mathbb{Z}$.

If we multiply both sides by 6 this gives us

$$30p = 6^{k+1} - 6$$

and adding 5 to both sides

$$30p + 5 = 6^{k+1} - 1$$

or

$$5(6p+1) = 6^{k+1} - 1$$

But this means $5 \mid (6^{k+1} - 1)$, since $6p+1$ is an integer.

Hence it's true for $n=1$, and when it's true for $n=k$ it must also be true for $n=k+1$, so by induction it's true for all $n \in \mathbb{N}$. $\square$