Examlet 2c **Foundations of Advanced Math** **2/23/07**

1. a) If A and B are sets, state the definition of $A \cap B$.

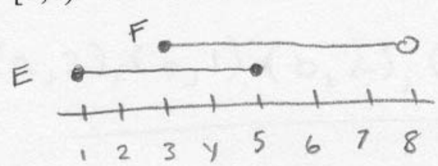
$$A \cap B = \{x \mid x \in A \land x \in B\}$$

Good

b) Let $C = \{1,2,3\}$ and $D = \{3,4,5\}$. What is $C \cup D$?

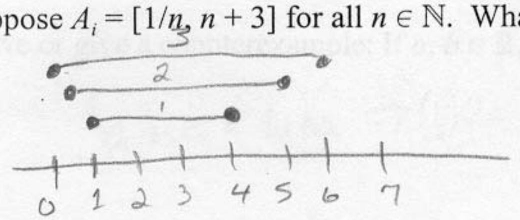
$$C \cup D = \{1, 2, 3, 4, 5\}$$

c) Let $E = [1,5]$ and $F = [3,8)$. What is $E - F$?



$$E - F = [1, 3)$$

2. a) Suppose $A_i = [1/n, n + 3]$ for all $n \in \mathbb{N}$. What is $\bigcup_{n\in\mathbb{N}} A_n$?



$\bigcup_{n\in N} A_n = (0, \infty)$

b) Let $A_i = [1/n, n + 3]$ for all $n \in \mathbb{N}$ as in part a. What is $\bigcap_{n\in\mathbb{N}} A_n$?

$\bigcap_{n\in N} A_n = [1, 4]$

c) Let $B = \{a, b, c\}$ and $C = \{1, 2\}$. What is $B \times C$?

$B \times C = \{(a,1), (a,2), (b,1), (b,2), (c,1), (c,2)\}$

Good.

3. a) Prove or give a counterexample: If a, $b \in \mathbb{R}$, with $a < b$, then $a < \dfrac{a+b}{2} < b$.

$a < b \Rightarrow \frac{a}{2} < \frac{b}{2}$

adding both sides $\frac{a}{2}$.

$\Rightarrow \frac{a}{2} + \frac{a}{2} < \frac{b}{2} + \frac{a}{2} \Rightarrow a < \frac{a+b}{2}$. (1)

$\frac{a}{2} < \frac{b}{2}$

adding both sides $\frac{b}{2}$

$\frac{a}{2} + \frac{b}{2} < \frac{b}{2} + \frac{b}{2}$

$\Rightarrow \frac{a+b}{2} < b$. (2)

Combine (1) and (2) $\Rightarrow a < \frac{a+b}{2} < b$

Nice

b) Prove or give a counterexample: If $a, b, c, d \in \mathbb{R}$, with $a < b$ then $\sqrt{ab} \le \dfrac{a+b}{2}$.

Counter examples

$a = -1$

$b = -2$.

$\sqrt{ab} = \sqrt{(-1)(-2)} = \sqrt{2}$

$\frac{a+b}{2} = \frac{-1-2}{2} = \frac{-3}{2}$

} check $\sqrt{2}$ is not smaller than $-\frac{3}{2}$

$\Rightarrow$ it is not true. for all a, b.

Excellent!

4. Let $\{A_i \mid i \in I\}$ be an indexed family of sets, all subsets of some universal set. Show that

$$\left(\bigcup_{i\in I} A_i\right)' = \bigcap_{i\in I} A_i' .$$

Well, first take $x \in \left(\bigcup_{i\in I} A_i\right)'$, so $x \notin \bigcup_{i\in I} A_i$. Then we must have $x \notin A_i$ for all $i \in I$, since otherwise if $x \in A_i$ for some $i \in I$ we'd have $x \in \bigcup_{i\in I} A_i$. But then since $x \notin A_i$ holds $\forall i \in I$, $x \in A_i'$ holds $\forall i \in I$, so $x \in \bigcap_{i\in I} A_i'$. Thus $\left(\bigcup_{i\in I} A_i\right)' \subseteq \bigcap_{i\in I} A_i'$.

Now take $x \in \bigcap_{i\in I} A_i'$, so $x \in A_i'$ for all $i \in I$. Then $x \notin A_i$ for all $i \in I$, so it can't be that $x \in \bigcup_{i\in I} A_i$ because that would mean $\exists i_0 \in I \ni x \in A_{i_0}$, a contradiction. Therefore $x \notin \bigcup_{i\in I} A_i$, or $x \in \left(\bigcup_{i\in I} A_i\right)'$. Thus we have $\bigcap_{i\in I} A_i' \subseteq \left(\bigcup_{i\in I} A_i\right)'$, and by mutual inclusion $\left(\bigcup_{i\in I} A_i\right)' = \bigcap_{i\in I} A_i'$ as desired. $\square$

5. Let A, B, and C be sets. Show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

First take $x \in A \cup (B \cap C)$, so $x \in A$ or $x \in B \cap C$.

Case 1: $x \in A$. Then $x \in A \cup B$ and $x \in A \cup C$, so $x \in (A \cup B) \cap (A \cup C)$.

Case 2: $x \in B \cap C$, so $x \in B$ and $x \in C$. Then since $x \in B$, we have $x \in A \cup B$. Similarly since $x \in C$, $x \in A \cup C$. Then because $x \in A \cup B$ and $x \in A \cup C$, we have $x \in (A \cup B) \cap (A \cup C)$.

So in either case $x \in (A \cup B) \cap (A \cup C)$, so $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Now take $x \in (A \cup B) \cap (A \cup C)$, so $x \in A \cup B$ and $x \in A \cup C$. Since $x \in A \cup B$, we know $x \in A$ or $x \in B$.

Case 1: $x \in A$. Then $x \in A \cup (B \cap C)$.

Case 2: $x \in B$. There are two subcases, depending on whether $x \in A$ holds.

Subcase i: $x \in A$. Then $x \in A \cup (B \cap C)$.

Subcase ii: $x \notin A$. We still know $x \in A \cup C$, and if $x \notin C$ we'd have a contradiction, so $x \in C$. Then we have $x \in B$ and $x \in C$, or $x \in B \cap C$. It follows that $x \in A \cup (B \cap C)$.

So in all cases we have $x \in A \cup (B \cap C)$, and thus $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Then by mutual inclusion $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$, as desired. $\square$