Examlet 3b **Foundations of Advanced Math** **3/23/07**

1. a) State the definition of an increasing function

A function is said to be increasing iff when $x > y$, then $f(x) > f(y)$ for all x and y. Good.

b) Give an example of a set which is not countable.

$(0,1)$

Excellent!

Or if there exists a uncountable set A then $A - \{x\}$ is not countable as well.

I just like that. :)

2. Is the product of an even function with an odd function even or odd? Support your answer.

odd.

Let $f(x)$ be an even function where $f(x) = f(-x)$

Let $g(x)$ be an odd function where $-g(x) = g(-x)$

$$f(x) \cdot -g(x) = f(-x) \cdot g(-x)$$

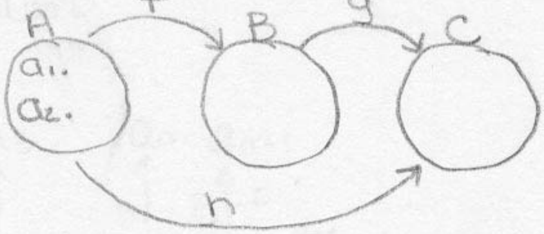
$$-fg(x) = fg(-x) \quad \text{Let the function } fg = h$$

$$-h(x) = h(-x)$$ which meets the definition of an odd function.

$\therefore$ The product of an even function w/ an odd function is an odd function

Great

3. Let $f: A \to B$ and $g: B \to C$ both be injections. Show that $g \circ f$ is injective.



Let a_1 and a_2 be elements of the set A, and

Let $h = g \circ f$.

So, suppose $h(a_1) = h(a_2)$

$$g \circ f(a_1) = g \circ f(a_2)$$

$$g(f(a_1)) = g(f(a_2)).$$

since g is an injection, then

$$f(a_1) = f(a_2)$$

and since f is an injection, then

$$a_1 = a_2$$

therefore h. is an injective, because no two elements of the set A can have the same answer. □

Excellent!

4. Suppose that A is a denumerable set, and let $B = \{1,2,3\}$. Prove that $A \times B$ is denumerable, or prove that it isn't.

So $A \times B$ consists of ordered pairs whose first elements are from A and whose second elements are from B, so for each $a \in A$ we have $(a, 1) \in A \times B$, $(a, 2) \in A \times B$, and $(a, 3) \in A \times B$.

To show $A \times B$ denumerable we need to provide a bijection $f : \mathbb{N} \to A \times B$. But we know that A is denumerable so $\exists$ a bijection $g : \mathbb{N} \to A$. Then our basic idea will be to send the first throdd, throddodd, and threeven to elements $(a_1, 1)$, $(a_1, 2)$, and $(a_1, 3)$, where $a_1 = f(1)$, and so forth.

The actual bijection is

$$g(n) = \begin{cases} (f(n), 1) & \text{if } n = 3k-2 \text{ for some } k \in \mathbb{N} \\ (f(n), 2) & \text{if } n = 3k-1 \text{ for some } k \in \mathbb{N} \\ (f(n), 3) & \text{if } n = 3k \text{ for some } k \in \mathbb{N} \end{cases}$$

Every element of $A \times B$ is in the range of g because every element of A is in the range of f and every element of B is paired with each. The function is injective because no outputs from different branches have the same second element and f is injective.

5. Show that for any $a, b \in \mathbb{R}$ with $a < b$, (a, b) and $\mathbb{R}$ are equipollent.

For two sets to be equipollent means there exists a bijection between them.

But on a problem set we showed $\mathbb{R}$ and $(0, 1)$ are equipollent, so if we can find a bijection from $(0, 1)$ to (a, b) then the composition of it with the bijection from the problem set will be the bijection we need.

So define $f : (0, 1) \to (a, b)$ by $f(x) = (b-a)x + a$, which is bijective since it's a non-constant linear function. $\square$

Scratch:

$(0, a)$ to $(1, b)$

$m = \frac{b-a}{1-0} = b-a$

$y - a = (b-a)(x-0)$

$y = (b-a)x + a$