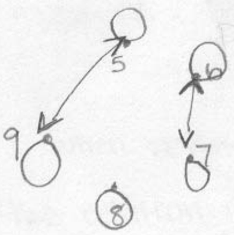
**Examlet 4a      Foundations of Advanced Math      4/13/07**

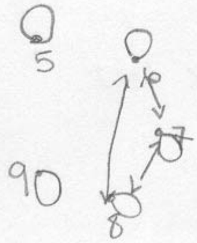
1. a) Let $A = \{5, 6, 7, 8, 9\}$ and $R_a = \{(5,5), (5,9), (6,6), (6,7), (7,6), (7,7), (8,8), (9,5), (9,9)\}$. What partition $P_a$ corresponds to the relation $R_a$ on $P_a$?



$P_a = \{\{5,9\}, \{6,7\}, \{8\}\}$

Good

b) Let $A$ be as above, and let $P_b = \{\{5\}, \{6,7,8\}, \{9\}\}$. What relation corresponds to the partition $P_b$?



$R_b = \{(5,5), (6,6), (6,7), (6,8), (7,6), (7,7), (7,8), (8,6), (8,7), (8,8), (9,9)\}$

Yes

2. a) Give an example of a relation on the set $\{1,2,3,4,5\}$ that is reflexive, symmetric, and transitive.

$R = \{(1,1), (2,2), (3,3), (4,4), (5,5)\}$

Good.

b) Give an example of a relation on the set $\{1,2,3,4,5\}$ that is reflexive, not symmetric and not transitive.

$R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (1,2), (2,3)\}$

Excellent

Not symmetric because it does not contain (2,1) & (3,2)

Not transitive because it does not contain (1,3)

3. Consider the relation ◂ on $\mathbb{R}$ defined by $x ◂ y \Leftrightarrow x \le 2y$. Determine whether ◂ is reflexive, symmetric, or transitive, and justify your conclusions clearly.

reflexive $x ◂ x$ $x \le 2x$ False, let $x = -1$

Counterexample

Good!

$-1 \le -2$ is false, not reflexive

symmetric $x ◂ y \rightarrow y ◂ x$ use $x = 3$, $y = 20$

Counterexample

$x \le 2y$

$3 \le 40 \rightarrow$ True does it hold for $y ◂ x$

$40 \le 3 \rightarrow$ Not true, the relation is

Great

not symmetric

transitive

Counterexample

$x ◂ y \wedge y ◂ z \rightarrow x ◂ z$

$x = 1$
$y = .6$
$z = .4$

$x \le 2y$ $y \le 2z$

$1 \le 2(.6)$ $.6 \le 2(.4)$

$1 \le 1.2$ $.6 \le .8$

Now try

$x \le 2z$

$1 \le 2(.4)$

$1 \le .8 \rightarrow$ Not true,

the relation is

not transitive.

Excellent

4. Let $\mathcal{F}$ be a partition of a set $A$. Define a relation $R$ on $A$ by

$$(a,b) \in R \Leftrightarrow (\exists X \in \mathcal{F})\,[a, b \in X]$$

Show that $R$ is an equivalence relation on $A$.

Well, to be an equivalence relation it has to be reflexive, symmetric, and transitive.

Reflexive: Since $\mathcal{F}$ is a partition we know $\bigcup \mathcal{F} = A$, so if we take an arbitrary $a \in A$ we know $a \in X$ for some $X \in \mathcal{F}$. Then $a \in X$ and $a \in X$, so $(a,a) \in R$ and our relation is reflexive.

Symmetric: S'pose $(a,b) \in R$, so $\exists X \in \mathcal{F}$ for which $a \in X$ and $b \in X$. Then we also have $(b,a) \in R$ and $R$ is symmetric.

Transitive: S'pose $(a,b) \in R$ and $(b,c) \in R$, so $\exists X_1 \in \mathcal{F}$ with $a, b \in X_1$ and $\exists X_2 \in \mathcal{F}$ with $b, c \in X_2$. But we know the sets in our partition are disjoint, so since $b \in X_1$ and $b \in X_2$, it must be that $X_1 = X_2$. Then $a$ and $c$ are both in this same set, and thus $(a,c) \in R$ and $R$ is transitive.

So since $R$ is reflexive, symmetric, and transitive, it's an equivalence relation, as desired. $\square$

5. a) Regarding the function $f : A \to B$ as a subset of $A \times B$, write the definition of $f^{-1}$.

$$f^{-1} = \{(y, x) \mid (x, y) \in f\}$$

given f is a bijection.

$x \in A$ and
$y \in B$.

Great

b) Let $A$ be some set. Write the identity function $i: A \to A$ as a relation on $A \times A$.

$$i = \{(a, a) \mid a \in A\}$$

Excellent