Examlet 4b **Foundations of Advanced Math** **4/13/07**

1. a) Let $A = \{5, 6, 7, 8, 9\}$ and $R_a = \{(5,5), (5,6), (5,7), (6,5), (6,6), (6,7), (7,5), (7,6), (7,7), (8,8), (9,9)\}$. What partition P_a corresponds to the relation R_a on P_a?

$[5] = \{5, 6, 7\}$
$[6] = \{6, 5, 7\}$
$[7] = \{7, 5, 6\}$
$[8] = \{8\}$
$[9] = \{9\}$

$P_a = \{\{5,6,7\}, \{8\}, \{9\}\}$ Good

b) Let A be as above, and let $P_b = \{\{5,8\}, \{6,7,9\}\}$. What relation corresponds to the partition P_b?

$R_b = \{(5,5), (5,8), (8,5), (8,8), (6,6), (7,7), (9,9), (6,7), (6,9), (7,6), (7,9), (9,6), (9,7))$

Great

2. a) Give an example of a relation on the set $\{1,2,3,4,5\}$ that is reflexive, symmetric, and transitive.

$R = \{(1,1)(2,2), (3,3), (4,4), (5,5)\}$ Smart

Is reflexive because $(\forall a \in R)[aRa]$

Is symmetric because $(\forall a,b \in R)[aRb \Rightarrow bRa]$

Is transitive because $(\forall a,b,c \in R)[aRb \wedge bRc \Rightarrow aRc]$

b) Give an example of a relation on the set $\{1,2,3,4,5\}$ that is reflexive and symmetric but not transitive.

$R = \{(1,1), (2,2), (3,3)(4,4), (5,5), (1,3), (3,5), (3,1), (5,3)\}$

Is reflexive b/c $(\forall a \in R)[aRa]$

Is symmetric b/c $(\forall a, b \in R)[aRb \Rightarrow bRa]$

Wonderful

Is Not transitive b/c $\{1R3 \wedge 3R5 \Rightarrow 1R5\}$ for it to be transitive but $(1,5)$ is not included in the relation.

3. Consider the relation ▸ on $\mathbb{R}$ defined by $x \blacktriangleright y \Leftrightarrow x \geq y + 1$. Determine whether ▸ is reflexive, symmetric, or transitive, and justify your conclusions clearly.

Reflexive ? $x \blacktriangleright x$ $\quad x \geq x + 1$

No, $2 \overset{?}{\geq} 2 + 1$ $\quad 2 \not\geq 3$ Good! Not reflexive

Symmetric ? $x \blacktriangleright y \longrightarrow y \blacktriangleright x$

$x \geq y + 1$ $\quad y \leq x - 1$ $\quad -y \geq -x + 1$ Not symmetric

let $x = 2$, $y = 1$ $\quad 2 \geq 2$ is true $\quad 1 \geq 2 + 1$ False Great

Transitive ? $x \blacktriangleright y \wedge y \blacktriangleright z \longrightarrow x \blacktriangleright z$

$x \geq y + 1 \quad \wedge \quad y \geq z + 1$

$x + y \geq y + z + 2$

$-y \quad -y$

$x \geq z + 2$ $\quad$ if $x \geq z + 2$ then $x \geq (z + 1) + 1$

thus $x \geq z + 1$ $\quad$ Transitive

Nice!

4. Let $\mathcal{F}$ be a partition of a set A. Define a relation R on A by

$$(a,b) \in R \Leftrightarrow (\exists\, X \in \mathcal{F})\ [a, b \in X]$$

Show that R is an equivalence relation on A.

Well, to be an equivalence relation it has to be reflexive, symmetric, and transitive.

Reflexive: Since $\mathcal{F}$ is a partition we know $\bigcup \mathcal{F} = A$, so if we take an arbitrary $a \in A$ we know $a \in X$ for some $X \in \mathcal{F}$. Then $a \in X$ and $a \in X$, so $(a,a) \in R$ and our relation is reflexive.

Symmetric: S'pose $(a,b) \in R$, so $\exists X \in \mathcal{F}$ for which $a \in X$ and $b \in X$. Then we also have $(b,a) \in R$ and R is symmetric.

Transitive: S'pose $(a,b) \in R$ and $(b,c) \in R$, so $\exists X_1 \in \mathcal{F}$ with $a, b \in X_1$ and $\exists X_2 \in \mathcal{F}$ with $b, c \in X_2$. But we know the sets in our partition are disjoint, so since $b \in X_1$ and $b \in X_2$, it must be that $X_1 = X_2$. Then a and c are both in this same set, and thus $(a,c) \in R$ and R is transitive.

So since R is reflexive, symmetric, and transitive, it's an equivalence relation, as desired. $\square$

5. a) Regarding the functions $f : A \to B$ and $g : B \to C$ as subsets of $A \times B$ and $B \times C$, respectively, write a definition of the composition function $g \circ f$.

$$g \circ f = \{ (a,c) \in A \times C \mid (a,b) \in f \wedge (b,c) \in g \}$$

b) Let A and B be sets. Write the definition of a constant function $h : A \to B$ as a set of ordered pairs.

$h : A \to B$ is a <u>constant</u> <u>function</u> iff

$$h = \{ (a, b_0) \mid a \in A \}$$

for some fixed $b_0 \in B$.