Exam 2 Calc 2 2/27/2008

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Show that $\int x\cos x\,dx = x\sin x + \cos x + C$.

$$\int x\cos x\,dx = x\sin x + \cos x + C$$

Integration by parts

$u = x \qquad v = \sin x$

$u' = 1 \qquad v' = \cos x$

$$= x\sin x - \int 1\cdot \sin x\,dx$$

$$= x\sin x - \int \sin x\,dx$$

$$= x\sin x - (-\cos x)$$

$$\boxed{= x\sin x + \cos x + C}$$

Great!

2. State the formula for the surface area obtained by rotating the curve $y = f(x)$, $a \le x \le b$, about the x-axis [assuming that $y = f(x)$ is positive for all values of x].

$y = f(x)$

$y' = f'(x)$

$$S.A. = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$$

$$S.A. = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

Good!

3. Evaluate $\int \sin^3 \theta \cos^2 \theta \, d\theta$.

$\int \sin\theta \cdot \sin^2\theta \cdot \cos^2\theta \, d\theta$

$\int \sin\theta (1-\cos^2\theta) \cos^2\theta \, d\theta$

$\int \cos^2\theta \sin\theta \, d\theta - \int \cos^4\theta \sin\theta \, d\theta$

let $u = \cos\theta$

$\frac{du}{d\theta} = -\sin\theta$

$\frac{du}{-\sin\theta} = d\theta$

$\int u^2 \cancel{\sin\theta} \frac{du}{\cancel{-\sin\theta}} + \int u^4 \cancel{\sin\theta} \frac{du}{\cancel{+\sin\theta}}$

$-\int u^2 du + \int u^4 du$

$-\frac{1}{3}u^3 + \frac{1}{5}u^5 + C$

$-\frac{1}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta + C$

Excellent!

4. Set up and evaluate an integral for the arc length of the function $f(x) = x^2/2$ between (0,0) and (2,4). [Hint: You can use the results of problems 8 and 9]

$f(x) = \frac{x^2}{2}$ $\quad 0 < x < 2$

$f'(x) = x$

Well done

$$\text{Arclength} = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

$$= \int_0^2 \sqrt{1+[x]^2}\,dx$$

$$= \int_0^2 \sqrt{1^2+x^2}\,dx \quad \text{(see line 21)}$$

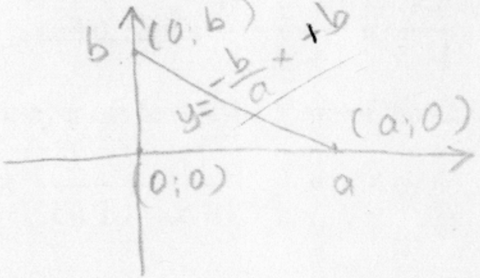
$$= \left[\frac{x}{2}\sqrt{1+x^2} + \frac{1}{2}\ln\left(x+\sqrt{1+x^2}\right)\right]_0^2$$

$$= \left[\frac{2}{2}\sqrt{1+4} + \frac{1}{2}\ln\left(2+\sqrt{1+4}\right)\right] - \left[\frac{0}{2}\sqrt{1+0} + \frac{1}{2}\ln\left(0+\sqrt{1+0}\right)\right] =$$

$$\left[\sqrt{5} + \frac{1}{2}\ln\left(2+\sqrt{5}\right)\right] - [0+0]$$

$$= \boxed{\sqrt{5} + \frac{1}{2}\ln\left(2+\sqrt{5}\right)}$$

5. Show that if a region shaped like a right triangle with legs of length a and b is positioned so that the right angle is at the origin, the leg of length a lies along the positive x-axis, and the leg of length b lies along the y-axis, then $\overline{x}$, the x coordinate of the center of mass, lies at $a/3$.



$$\overline{x} = \frac{\int_0^a x\cdot\left(\frac{-b}{a}x + b\right)dx}{\int_0^a \left(\frac{-b}{a}x + b\right)dx}$$

Nice Job!

$$= \frac{\left(\frac{-b}{3a}x^3 + \frac{b}{2}x^2\right)\Big|_0^a}{\left(\frac{-b}{2a}x^2 + bx\right)\Big|_0^a} = \frac{\frac{-b\cdot a^3}{3a} + \frac{b\cdot a^2}{2}}{\frac{-b\cdot a^2}{2a} + b\cdot a} = \frac{\frac{a^2b}{6}}{\frac{ab}{2}} = \boxed{\frac{a}{3}}$$

6. The function $f(t)=\begin{cases} 0 & \text{if } t<0 \\ t\,e^{-t} & \text{if } t \geq 0 \end{cases}$ is a probability density function. Compute the mean for this p.d.f.

By line 97

$$\bar{z} = \frac{\int_{-\infty}^{\infty} t\cdot f(t)\,dt}{\int_{-\infty}^{\infty} f(t)\,dt} = \frac{\int_0^{\infty} t\cdot e^{-t}\,dt}{1} = \lim_{b\to\infty}\left[-t^2e^{-t} + 2\int te^{-t}\,dt\right]_0^b$$

Because it's a p.d.f.

By line 96

$$= \lim_{b\to\infty}\left[-t^2e^{-t} + 2(-t-1)e^{-t}\right]_0^b = \lim_{b\to\infty}\left(-\frac{b^2}{e^b} - \frac{2b}{e^b} - \frac{2}{e^b} - \frac{-2}{1}\right)$$

$$\overset{L'H}{=} \lim_{b\to\infty}\left(-\frac{2b}{e^b} - \frac{2}{e^b} + 2\right) \overset{L'H}{=} \lim_{b\to\infty}\left(\frac{-2}{e^b} + 2\right) = 2$$

7. Biff is a calculus student at Enormous State University, and he has a question. Biff says "Dude, I'm cramming for my calc test, and I think these partial fraction things are really whacked. The test from last year I paid my frat brother \$50 for has this question with, like, $\frac{x^3}{x(x+1)(x-1)}$, and I did the stuff and got ½ and -½ and 0, so it's $\frac{0}{x}+\frac{-\frac{1}{2}}{x+1}+\frac{\frac{1}{2}}{x-1}$, right? But then I looked at the back of the book, and they must have used some of that crazy log property stuff, 'cause they got the integral to be $x+\frac{1}{2}\ln\left(\frac{|x-1|}{|x+1|}\right)+C$. How'd they do that?"

Help Biff by pointing out any issues with his approach, or suggesting how to make his result match the book's.

Biff, I don't think it's the log properties that got you. It looks like you started off with your partial fractions decomposition a little bit too soon. Notice how the degrees of both the numerator and denominator in your original function are both three? So that means you have to do division first, which I bet produces a "1" that leads to the "x" part of their answer. There's a log property at the end too, probably, to put $\ln|x-1|$ and $-\ln|x+1|$ together to make $\ln\left|\frac{x-1}{x+1}\right|$, but your real mistake was not paying attention at the beginning!

8. Show that $\int\sqrt{a^2+u^2}\,du$ can be transformed by an appropriate substitution into $\int a^2\sec^3\theta\,d\theta$.

$\sin^2+\cos^2=1$

$\tan^2+1=\sec^2$

$\int\sqrt{a^2+u^2}\,du$ let $u=a\tan\theta$

$\frac{du}{d\theta}=a\sec^2\theta$

$du=a\sec^2\theta\,d\theta$

$\int\sqrt{a^2+a^2\tan^2\theta}\cdot a\sec^2\theta\,d\theta$

$a^2\int\sqrt{1+\tan^2\theta}\cdot\sec^2\theta\,d\theta$

$a^2\int\sqrt{\sec^2\theta}\cdot\sec^2\theta\,d\theta$

Great

$a^2\int\sec\theta\cdot\sec^2\theta\,d\theta$

$a^2\int\sec^3\theta\,d\theta = \boxed{\int a^2\sec^3\theta\,d\theta}$

9. Derive line 77 from our table of integrals.

$$\int \sec^n u \, du = \int \sec^{n-2} u \cdot \sec^2 u \, du \qquad \text{let } u = \sec^{n-2} u \qquad v = \tan u$$

$$u' = (n-2)\sec^{n-3} u \cdot \sec u \cdot \tan u \qquad v' = \sec^2 u$$

$$= \sec^{n-2} u \cdot \tan u - \int \tan u \cdot (n-2) \cdot \sec^{n-2} u \tan u \, du$$

$$= \sec^{n-2} u \cdot \tan u - (n-2) \int \tan^2 u \sec^{n-2} u \, du$$

$$= \sec^{n-2} u \tan u - (n-2) \int (\sec^2 u - 1) \sec^{n-2} u \, du$$

$$= \sec^{n-2} u \tan u - (n-2) \int \sec^n u \, du + (n-2) \int \sec^{n-2} u \, du$$

$$(n-1) \int \sec^n u \, du = \sec^{n-2} u \tan u + (n-2) \int \sec^{n-2} u \, du$$

$$\int \sec^n u \, du = \frac{1}{n-1} \tan u \sec^{n-2} u + \frac{n-2}{n-1} \int \sec^{n-2} u \, du \qquad \square$$

10. Evaluate $\displaystyle\int\left(\frac{13}{(x^2+4)(x-3)}\right)dx$. Partial Fractions!

I wish: $\dfrac{13}{(x^2+4)(x-3)} = \dfrac{Ax+B}{x^2+4} + \dfrac{C}{x-3}$

$$13 = (Ax+B)(x-3) + C(x^2+4)$$

If $x=3$:

$13 = 0(Ax+B) + 13C$

$\therefore C = 1$

If $x=0$:

$13 = -3(A\cdot 0+B) + 4C$

$13 = -3\cdot 0 - 3B + 4$

$9 = -3B$

$\therefore B = -3$

If $x=1$:

$13 = (A+B)\cdot -2 + 5C$

$13 = -2A - 2(-3) + 5(1)$

$13 = -2A + 6 + 5$

$2 = -2A$

$\therefore A = -1$

So $\displaystyle\int \frac{13}{(x^2+4)(x-3)}\,dx = \int\left(\frac{-1x-3}{x^2+4} + \frac{1}{x-3}\right)dx$

$$= \int \frac{-x}{x^2+4}\,dx + \int \frac{-3}{x^2+4}\,dx + \int \frac{1}{x-3}\,dx$$

let $u = x^2+4$

$\frac{du}{dx} = 2x$

$\frac{du}{2x} = dx$

$$= \int \frac{-\cancel{x}}{u}\cdot\frac{du}{2\cancel{x}} - 3\left(\tfrac{1}{2}\arctan\tfrac{x}{2}\right) + \ln|x-3| + C$$

$$= \frac{-1}{2}\ln|x^2+4| - \frac{3}{2}\arctan\frac{x}{2} + \ln|x-3| + C$$