Exam 4 Calc 2 4/18/2008

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Find the first 3 partial sums for the series $\sum_{n=1}^{\infty} \frac{1}{n}$.

$\boxed{S_1 = 1}$

$S_2 = 1 + \frac{1}{2} = \boxed{\frac{3}{2}}$

$S_3 = 1 + \frac{1}{2} + \frac{1}{3} = \boxed{\frac{11}{6}}$

Good!

2. Give an example of a series which converges, but does not converge absolutely.

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, since it converges (we know it as the alternating harmonic series) but $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right|$ diverges (because we know that about the harmonic series).

3. Determine whether $\sum_{n=1}^{\infty} \frac{3n}{n^3+1}$ converges or diverges.

Use Limit Comparison Test!

compare to $\frac{1}{n^2}$ which converges by the p-series w/ $p>1$

So, $\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{3n}{n^3+1}} = \lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{n^3+1}{3n} = \lim_{n\to\infty} \frac{n^3+1}{3n^3} \overset{\frac{\infty}{\infty}\ L'H}{=} \lim_{n\to\infty} \frac{3n^2}{9n^2} \overset{\frac{\infty}{\infty}\ L'H}{=}$

$\lim_{n\to\infty} \frac{6n}{18n} \overset{\frac{\infty}{\infty}\ L'H}{=} \lim_{n\to\infty} \frac{6}{18} = \frac{1}{3}$ and because $\frac{1}{n^2}$ converges, and we got a finite #, then by the Limit Comparison Test this series converges!

Nice Job!

10

4. Determine whether $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ converges or diverges.

Use Integral Test!

$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x\sqrt{\ln x}}\,dx$

let $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$dx = du \cdot x$

$= \lim_{b\to\infty} \int_2^b \frac{1}{\cancel{x}\sqrt{u}} \cdot \cancel{x}\,du = \lim_{b\to\infty} \int_2^b u^{-1/2}$

Well done!

$= \lim_{b\to\infty} \left[2(\ln x)^{1/2}\right]_2^b = \lim_{b\to\infty} \left[(2(\ln b)^{1/2}) - (2(\ln 2)^{1/2})\right] = \infty$

So, the series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ diverges because $\int \frac{1}{x\sqrt{\ln x}}\,dx$ diverges by the Integral Test!

5. Find the radius of convergence of the series $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$.

Use Rat. Test!

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1} x^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{(-1)^n x^{2n+1}}{(2n+1)!}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{\cancel{n+1}} \cancel{x}^{\cancel{2n}+3}}{(2n+3)!} \cdot \frac{\cancel{(2n+1)!}}{\cancel{(-1)^n} \cancel{x^{2n+1}}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{-1 \cdot x^2}{(2n+3)(2n+2)}\right|$$

$$= \lim_{n\to\infty} |-x^2| \cdot \frac{1}{(2n+3)(2n+2)} \quad 1/\infty = 0$$

Excellent!

$= \underline{0}$

Thus, this series converges no matter what the value x is. The radius of convergence for this series is $\underline{\infty}$ (infinity) because 0 < 1, series will always converge.

$(x^3)^2$

6. Find a power series, in sigma notation, for $f(x) = \dfrac{1}{1+x^3}$

I know: $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$

So.

$$\frac{1}{1-(-x^3)} = 1 - x^3 + x^2 - x^3 + x^4 + \dots$$

Therefore

$$\frac{1}{1+(x^3)} = 1 - x^3 + x^6 - x^9 + x^{12} + \dots$$

And Finally: $\boxed{\sum_{n=0}^{\infty} (-1)^n x^{3n}}$

Excellent!

Checking: $(-1)^1 \cdot x^{3(1)} = -x^3 + x^6 - x^9$ ✓

$(-1)^2 \cdot x^{3(2)}$

$(-1)^3 \cdot x^{(3)(3)}$

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod, this is *so* unfair. Our professor was talking about this thing, like, with a lion chasing a guy around in this arena, right? And he wrote all this stuff on the board, and I didn't understand *any* of it, but at the end, like, there was one of those sigma things and one over n to the three-quarters. But so then he said we had to know what that was for our test, because, like, it was how far the guy could run before the lion would eat him. But he also told us the day before that pretty much the only of those sigma things where we could actually say what they added up to were the geometric ones, and I don't think this is like that. So how am I ever going to pass this test?"

Help Bunny by explaining to her what the series she mentions tells her about the man's life expectancy.

Bunny. I can understand how you are so confused. Just reading this confused me. Now, when you have a series like $\Sigma \frac{1}{n^{3/4}}$, you have to consider all of your possibilities. First, your professor wants you to figure out if this series converges or diverges. If it converges, then the guy is going to be eaten by the lion at the place it converges to, and if it diverges, then the man will run and run forever and the lion will never catch him. If you remember one of the very first days learning about series, then you should remember p-series $= \frac{1}{x^p}$, if $p > 1$ it converges, but if $p \leq 1$, then it diverges. Your series $\Sigma \frac{1}{n^{3/4}}$ is a p-series w $p \leq 1$, so it diverges. With that said, the man will be able to outrun the lion forever! Yay!

10

8. Find the 3rd degree Maclaurin Polynomial for $f(x) = \tan x$.

$f(x) = \tan(x)$ $\qquad f(0) = 0$

$f'(x) = \frac{1}{\cos(x)^2}$ $\qquad f'(0) = 1$

$f''(x) = \frac{2\sin(x)}{\cos(x)^3}$ $\qquad f''(0) = 0$

$f'''(x) = \frac{2(2\sin x^2 + 1)}{\cos x^4}$ $\qquad f'''(0) = 2$

$$\sum_{1}^{3} \frac{f^n(0)}{n!} x^n$$

$$\frac{f(0)}{0!}x^0 + \frac{f'(0)}{1!}x^1 + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$0 + \frac{1}{1!}x^1 + \frac{0}{2!}x^2 + \frac{2}{3!}x^3$$

$1 \cdot 2 \cdot 3$

$$\boxed{x + \frac{1}{3}x^3}$$

Good

9. Determine whether $x = ½$ is included in the interval of convergence of the series $\sum_{n=1}^{\infty}(-1)^n \frac{2^n x^n}{n^2}$.

Plug in $x = \frac{1}{2}$:

$$\sum_{n=1}^{\infty}(-1)^n \frac{2^n \left(\frac{1}{2}\right)^n}{n^2} = \sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2}$$

Now use A.S.T. on this

Sign alternates? ✓

Limit zero?

$$\lim_{n\to\infty} \frac{1}{n^2} = 0 \checkmark$$

Decreasing?

$(n^{-2})' = -2n^{-3} = \frac{-2}{n^3}$, so numerator negative and denominator positive for all $n \geq 1$, so since the derivative is negative it's decreasing. ✓

So by A.S.T. it's convergent, so yes, $x = \frac{1}{2}$ is in the interval of convergence.

10. Find the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)} x^n$. Rat. Test!

$$\lim_{n\to\infty} \left| \frac{\frac{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)\cdot [2(n+1)]}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)\cdot [2(n+1)-1]} x^{n+1}}{\frac{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)} x^n} \right| = \lim_{n\to\infty} \left| \frac{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)\cdot (2n+2)}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)(2n+1)} \cdot \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)} \frac{x^{n+1}}{x^n} \right|$$

$$= \lim_{n\to\infty} \frac{2n+2}{2n+1} |x|$$

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{2}{2} |x|$$

$$= |x|$$ So it converges when $|x| < 1$.

Endpoints?

When $x = 1$:

The numerator is larger than the denominator, so

$$\lim_{n\to\infty} \frac{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)} \geq 1 \neq 0$$

and thus the series diverges by the Test for Divergence.

When $x = -1$:

What happened for $x=1$ happens again. That limit is still non-zero, so by the Test for Divergence it diverges.

$\therefore$ The interval of convergence is $(-1, 1)$.