Our analysis of the Hermite's equation can be compared to our study of constant-coefficient, second-order equations such as

$$\frac{d^2y}{dt^2} + n^2 y = 0.$$

The solutions of this constant-coefficient equation are linear combinations of $\sin nt$ and $\cos nt$. The Hermite polynomials share many properties with the functions $\sin nt$ and $\cos nt$, and both families of functions appear frequently in applications. Legendre's equation, another equation with similar properties, is studied in Exercise 15.

EXERCISES FOR APPENDIX B

In Exercises 1–4, use the guess-and-test method to find the power series expansion centered at $t = 0$ for the general solution up to degree four, that is, up to and including the t^4 term. (You may find the general solution using other methods and then find the Taylor series centered at $t = 0$ to check your computation if you like.)

1. $\dfrac{dy}{dt} = y$

2. $\dfrac{dy}{d2} = -y + 1$

3. $\dfrac{dy}{dt} = -2ty$

4. $\dfrac{dy}{dt} = t^2 y + 1$

In Exercise 5–8, find the power series expansion for the general solution up to degree four, that is, up to and including the t^4 term.

5. $\dfrac{dy}{dt} = -y + e^{2t}$

6. $\dfrac{dy}{dt} = 2y + \sin t$

7. $\dfrac{d^2y}{dt^2} + 2y = \cos t$

8. $\dfrac{d^2y}{dt^2} + 5\dfrac{dy}{dt} + y = \sin 2t$

9. Verify that $y(t) = \tan t$ is a solution of

$$\frac{dy}{dt} = y^2 + 1,$$

and compute a power series solution to find the terms up to degree six (up to and including the t^6 term) of the Taylor series centered at $t = 0$ of $\tan t$.

In Exercises 10–13, find the general solution up to degree six, that is, up to and including the t^6 term.

10. $\dfrac{d^2y}{dt^2} + 2y = 0$

11. $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = 0$

12. $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + t^2 y = \cos t$

13. $\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + y = e^{-2t}$

1. The Taylor series centered at $t = 0$ for $y(t) = ke^t$.

3. The Taylor series centered at $t = 0$ for $y(t) = ke^{-t^2}$.

5. $y(t) = a_0 + (-a_0 + 1)t + (a_0/2 + 1/2)\,t^2 + (-a_0/6 + 1/2)\,t^3 + (a_0/24 + 5/24)t^4 + \ldots$

7. $y(t) = a_0 + a_1 t + (1/2 - a_0)\,t^2 - (a_1/3)\,t^3 + (a_0/6 - 1/8)\,t^4 + \ldots$

9. $\tan t = t + t^3/3 + 2t^5/15 + \ldots$

11. $y(t) = a_0 + a_1 t + (-a_0/2 - a_1/2)\,t^2 + (a_0/6)t^3 + (a_1/24)t^4 + (-a_0/120 - a_1/120)\,t^5 + (a_0/720)t^6 + \ldots$

13. $y(t) = a_0 + a_1 t + (1/2 - a_0/2)\,t^2 + (-1/3 - a_1/3)\,t^3 + (1/24 + a_0/8)\,t^4 + (a_1/15)t^5 + (11/720 - a_0/48)\,t^6 + \ldots$

15. **(a)**

$$a_2 = -\frac{\nu(\nu+1)}{2}a_0,$$

$$a_3 = \frac{2 - \nu(\nu+1)}{2}a_1,$$

$$a_4 = -\frac{6 - \nu(\nu+1)}{12}\,\frac{\nu(\nu+1)}{2}a_0$$

(b) *Hint*: Note that a_{2n} has a_0 as a factor and a_{2n+1} has a_1 as a factor. Also note that if $\nu = n$ is a positive integer, then $a_{n+2} = 0$.

(c) *Hint*: Use the formulas from part (a).

(d) $P_3(t) = t - \frac{5}{3}t^3$,

$P_4(t) = 1 - 10t^2 + \frac{35}{3}t^4$,

$P_5(t) = t - \frac{14}{3}t^3 + \frac{21}{5}t^5$,

$P_6(t) = 1 - 21t^2 + 63t^4 - \frac{231}{5}t^6$

(e) *Hint*: Use linearity.

17. $y(t) = t - t^2 + t^3/2 - t^4/6 + \ldots$