**Exam 2a  Differential Equations  3/7/08**

Each problem is worth 10 points. For full credit indicate clearly how you reached your answer.

1. Determine whether $x(t) = 2e^{4t} - 6e^{t}, y(t) = 2e^{4t} + 3e^{t}$ is a solution to the system

$\frac{dx}{dt} = 8e^{4t} - 6e^{t}$

$\frac{dy}{dt} = 8e^{4t} + 3e^{t}$

$$\frac{dx}{dt} = 2x + 2y$$

$$\frac{dy}{dt} = 1x + 3y$$

$$\frac{dx}{dt} = 8e^{4t} - 6e^{t} = 2(2e^{4t} - 6e^{t}) + 2(2e^{4t} + 3e^{t})$$
$$= 4e^{4t} - 12e^{t} + 4e^{4t} + 6e^{t}$$
$$= 8e^{4t} - 6e^{t} \checkmark$$

$$\frac{dy}{dt} = 8e^{4t} + 3e^{t} = 1(2e^{4t} - 6e^{t}) + 3(2e^{4t} + 3e^{t})$$
$$= 2e^{4t} - 6e^{t} + 6e^{4t} + 9e^{t}$$
$$= 8e^{4t} + 3e^{t} \checkmark$$

Great!

yes, it is a solution.

2. State (you don't need proof) the Laplace transforms:

a) $\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a}$ for $s > a$

Good

b) $\mathcal{L}\left[u_a\right] = \frac{e^{-as}}{s}$ for $s > 0$

3. Consider the system
$$\frac{dx}{dt} = 10(y-x)$$
$$\frac{dy}{dt} = 28x - y - xz$$
$$\frac{dz}{dt} = -\tfrac{8}{3}z + xy$$
Find all equilibria of the system.

$0 = 10(y-x) \leftarrow$ so $\underline{x=y}$ to make this true

$0 = 28x - y - xz$

$\downarrow$

$xz = 27x \leftarrow$ either $\underline{x=0}$, or $\underline{z=27}$

$0 = -\frac{8}{3}z + xy$

$\downarrow$

$\frac{8}{3}z = x^2 \leftarrow$ if $x=0$, $\underline{z=0}$; if $z=27$, $\frac{8}{3}(27) = x^2$

$72 = x^2$

$\pm\sqrt{72} = x$

$\underline{\pm 6\sqrt{2} = x}$

$(0,0,0)$
$(6\sqrt{2}, 6\sqrt{2}, 27)$
$(-6\sqrt{2}, -6\sqrt{2}, 27)$

Excellent!

4. Write a system of differential equations to model two interacting populations (of creatures called *pinks* and *quinks*) represented by $p(t)$ and $q(t)$ with the following characteristics:

- pinks, in the absence of quinks, will experience exponential population decay.
- quinks, in the absence of pinks, have their population grow logistically to a carrying capacity of $Q$.
- interactions between pinks and quinks benefit the pink population.
- interactions between pinks and quinks hurt the quink population.

Use letters of your choice for proportionality constants, but write your system in such a way that all of these parameters have positive values.

$$\frac{dP}{dt} = -\underline{aP} + \underline{cPq}$$

Great!

$$\frac{dq}{dt} = \underline{bq\left(1-\frac{q}{Q}\right)} - \underline{dPq}$$

Q = carrying capacity

Constants

a = exponential pop decay constant

b = logistic growth constant

c = interaction benefit

d = interaction hurt.

5. Suppose that you know $x(t)=k_2e^{2t}-\frac{k_1}{3}e^{-t},\ y(t)=k_1e^{-t}$ is a general solution to a system of differential equations. Find the solution satisfying the initial condition $\mathbf{Y}(0)=(x(0),y(0))=(-1,3)$.

$3=k_1e^{(0)}$

$\underline{3=k_1}$

Good

$-1=k_2e^0-\frac{k_1}{3}e^0$

$-1=k_2-\frac{3}{3}$

$-1=k_2-1$

$\underline{k_2=0}$

$$\boxed{\begin{aligned} y(t)&=3e^{-t}\\ x(t)&=-1e^{-t}\end{aligned}}$$

6. Find a solution to the system $\begin{aligned}\frac{dx}{dt} &= x + 2y \\ \frac{dy}{dt} &= 3x\end{aligned}$.

$$\Rightarrow \frac{d^2x}{dt^2} = \frac{dx}{dt} + 2\frac{dy}{dt}$$

$$\frac{d^2x}{dt^2} = \frac{dx}{dt} + 2(3x)$$

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - 6x = 0$$

S'pose $x(t) = e^{st}$

$$\frac{dx}{dt} = s \cdot e^{st}$$

$$\frac{d^2x}{dt^2} = s^2 \cdot e^{st}$$

$$(s^2 \cdot e^{st}) - (s \cdot e^{st}) - 6(e^{st}) = 0$$

$$e^{st}(s^2 - s - 6) = 0$$

$$e^{st}(s-3)(s+2) = 0$$

$$\therefore s = 3 \text{ or } s = -2$$

So $x(t) = Ae^{3t} \Rightarrow \frac{dy}{dt} = 3Ae^{3t} \Rightarrow y(t) = Ae^{3t}$

Or $x(t) = Be^{-2t} \Rightarrow \frac{dx}{dt} = 3Be^{-2t} \Rightarrow y(t) = -\frac{3}{2}Be^{-2t}$

Hence $x(t) = e^{3t}$, $y(t) = e^{3t}$

and $x(t) = \cancel{\frac{3}{2}}e^{-2t}$, $y(t) = -\frac{3}{2}e^{-2t}$

are solutions.

or $x(t) = Ae^{3t} + Be^{-2t}$

$y(t) = Ae^{3t} - \frac{3}{2}Be^{-2t}$

is a general solution.

7. Show that $\mathcal{L}[t] = \dfrac{1}{s^2}$, and note any necessary restrictions.

$$\mathcal{L}[t] = \int_0^\infty t\, e^{-st}\, dt$$

$u = t \qquad v = -\frac{1}{s} e^{-st}$

$u' = 1\, dt \qquad v' = e^{-st}\, dt$

$$\mathcal{L}[t] = -\frac{t}{s} e^{-st} \Big|_0^\infty - \int_0^\infty -\frac{1}{s} e^{-st}\, dt$$

$$\mathcal{L}[t] = \lim_{b \to \infty} \frac{-t}{s} e^{-st} \Big|_0^b + \frac{1}{s}\left(-\frac{1}{s} e^{-st}\right) \Big|_0^b$$

$$\mathcal{L}[t] = \lim_{b \to \infty} \frac{-b}{s} e^{-s(b)} - \left(-\frac{0}{s} e^{-s(0)}\right) - \frac{1}{s^2} e^{-s(b)} + \frac{1}{s^2} e^{-s(0)}$$

L.H. $\lim_{b \to \infty} \dfrac{+1}{s^2 e^{-s(b)}} \to 0$ $\quad \to 0 \quad \to 0 \quad \to \frac{1}{s^2}$

$$\mathcal{L}[t] = 0 + 0 - 0 + \frac{1}{s^2}$$

$$\boxed{\mathcal{L}[t] = \frac{1}{s^2} \quad \text{for } s > 0}$$

Nice Job!

8. Compute the inverse Laplace transform $\mathscr{L}^{-1}\left[\frac{4}{s(s+3)}\right]$.

Partial Fractions

$$\frac{4}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}$$

$$As + 3A + Bs = 4$$

$$3A = 4$$

$$A = 4/3$$

$$A + B = 0$$

$$B = -A$$

$$B = -4/3$$

$$\mathscr{L}^{-1}\left[\frac{4}{s(s+3)}\right] = \mathscr{L}^{-1}\left[\frac{4/3}{s} + \frac{-4/3}{s+3}\right]$$

$$= \mathscr{L}^{-1}\left[\frac{4/3}{s}\right] + \mathscr{L}^{-1}\left[\frac{-4/3}{s+3}\right]$$

$$= \frac{4}{3}\mathscr{L}^{-1}\left[\frac{1}{s}\right] - \frac{4}{3}\mathscr{L}^{-1}\left[\frac{1}{s+3}\right]$$

$$= \frac{4}{3}\cdot 1 - \frac{4}{3}\cdot e^{-3t}$$

$$= \frac{4}{3}\left(1 - e^{-3t}\right)$$

Nice work!

9. Show that $\mathcal{L}\left[\frac{d^2y}{dt^2}\right] = s^2 \cdot \mathcal{L}[y] - s \cdot y(0) - y'(0)$.

Well, I know $\mathcal{L}\left[\frac{dy}{dt}\right] = s\mathcal{L}[y] - y(0)$

Let $x = \frac{dy}{dt}$, so $\mathcal{L}\left[\frac{d^2y}{dt^2}\right] = \mathcal{L}\left[\frac{dx}{dt}\right]$

$\frac{dx}{dt} = \frac{d^2y}{dt^2}$

$$\mathcal{L}\left[\frac{dx}{dt}\right] = s\,\mathcal{L}[x] - x(0)$$

$$= s\,\mathcal{L}\left[\frac{dy}{dt}\right] - \frac{dy}{dt}(0)$$

$$= s\left[s\mathcal{L}[y] - y(0)\right] - y'(0)$$

Beautiful!

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] = s^2\,\mathcal{L}[y] - s\cdot y(0) - y'(0) \quad \square$$

10. Suppose that you know that $\mathbf{Y}_1(t) = (x(t), y(t))$ is a solution to the system of equations

$$\frac{dx}{dt} = 3x - 5y$$
$$\frac{dy}{dt} = 2x + y$$

. Show that $\mathbf{Y}_2(t) = (a \cdot x(t), a \cdot y(t))$ is also a solution, for any real number $a$.

$$\frac{dx(t)}{dt} = 3(x(t)) - 5y(t) \qquad \frac{d(a \cdot x(t))}{dt} = 3(a \cdot x(t)) - 5(a$$

$$\frac{dy(t)}{dt} = 2(x(t)) - y(t) \qquad \frac{d(a \cdot y(t))}{dt} = 2(a \cdot x(t)) + a \cdot y(t)$$

Well since a is a constant, it can be pulled out.

$$a\left(\frac{dx(t)}{dt}\right) = a(3x(t) - 5y(t))$$

$$a\left(\frac{dy(t)}{dt}\right) = a(2x(t) + y(t))$$

Now we can see it is just like the other solution, multiplying by a constant does not change anything on the inside. As long as the constant is the same on both sides of the equal sign for both equations, it is legal to multiply through completely by the same constant. So it is a solution also.

Excellent!