Exam 3   Differential Equations   4/18/08

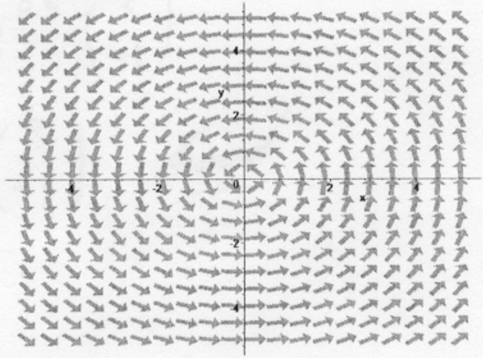
Each problem is worth 10 points. For full credit indicate clearly how you reached your answer.

1. One of the planar systems whose phase plane is shown at right has two distinct real nonzero eigenvalues, one has two purely imaginary eigenvalues, and one has two real eigenvalues, one of which is zero. Identify which is which.

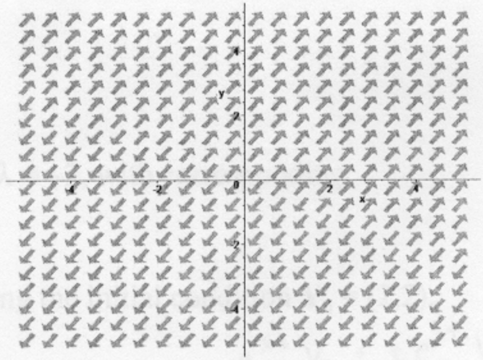
imaginary eigenvalues

complex =>

$\lambda = a \pm bi$



two real eigenvalues one of which is zero =>



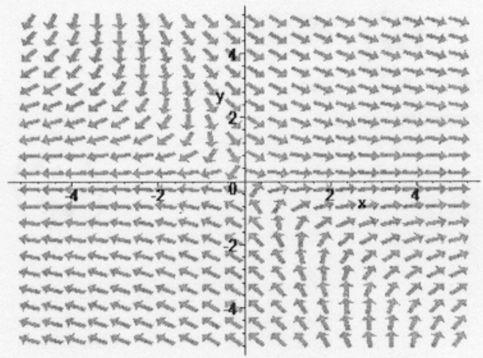
Good

saddle =>

real nonzero eigenvalues

$\lambda_1 =$ positive real

$\lambda_2 =$ negative real



2. Give an example of a planar linear system where the origin is a saddle.

For the origin to be a saddle, 1 eigenvalue is positive and one is negative, so

$(\lambda_1 < 0 < \lambda_2)$

$$\frac{dx}{dt} = x$$

$$\frac{dy}{dt} = -y$$

or $\frac{d\hat{Y}}{dt} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \hat{Y}$

Yes!

would work.

3. If a planar system of differential equations has eigenvalues $\lambda_1 = 3$, $\lambda_2 = 0$ and associated eigenvectors $\mathbf{v}_1 = (1,0)$ and $\mathbf{v}_2 = (3,-1)$, write a general solution to the system.

$$\hat{Y} = Ae^{3t}\begin{pmatrix}1\\0\end{pmatrix} + Be^{0 \cdot t}\begin{pmatrix}3\\-1\end{pmatrix}$$

$\hat{Y} = Ae^{3t}\begin{pmatrix}1\\0\end{pmatrix} + B\begin{pmatrix}3\\-1\end{pmatrix}$ is a general solution

Great

4. The system $\begin{aligned}\frac{dx}{dt} &= -2x - 1y \\ \frac{dy}{dt} &= 1x - 4y\end{aligned}$ has the single eigenvalue $\lambda = -3$, with corresponding eigenvector (1,1). Find a solution to this system satisfying the initial condition $\mathbf{Y}_0 = (2,2)$.

The Great theorem of pg. 305 says that when there are repeated eigenvalues, the solution is of the form $\hat{Y} = e^{\lambda t}\hat{V}_0 + te^{\lambda t}\hat{V}_1$ where $\hat{V}_0 = Y_0$ and $\hat{V}_1 = (\hat{A} - \lambda\hat{I})\hat{Y}_0$

So, $\hat{V}_1 = \begin{pmatrix} -2-\lambda & -1 \\ 1 & -4-\lambda \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\hat{Y} = e^{-3t}\begin{pmatrix} 2 \\ 2 \end{pmatrix} + te^{-3t}\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\hat{Y} = e^{-3t}\begin{pmatrix} 2 \\ 2 \end{pmatrix}$

Nice!

Excellent!

5. Find a general solution to the equation $y'' + 3y' - 4y = 0$.

maybe $y = e^{st}$, so $y' = se^{st}$, $y'' = s^2 e^{st}$

$$s^2 e^{st} + 3se^{st} - 4e^{st} = 0$$

$$e^{st}(s^2 + 3s - 4) = 0$$

$$e^{st}(s+4)(s-1) = 0$$

$$s = -4 \text{ or } s = 1$$

So, $y = K_1 e^{-4t} + K_2 e^{t}$

Good

6. Find a general solution to the equation $y'' + 3y' - 4y = e^{5t}$.

Solve homogeneous equation first: $y'' + 3y' - 4y = 0$
(which I did above) so $y_h = K_1 e^{-4t} + K_2 e^{t}$

now suppose $y_p = Ae^{5t}$, $y_p' = 5Ae^{5t}$, $y_p'' = 25Ae^{5t}$

Plugging it in, we get:

$$25Ae^{5t} + 15Ae^{5t} - 4Ae^{5t} = e^{5t}$$

$$36Ae^{5t} = e^{5t}$$

so $36A = 1$

$A = 1/36$

Excellent!

By Extended Linearity Principle:

$y_g = K_1 e^{-4t} + K_2 e^{t} + \frac{1}{36} e^{5t}$

is a general solution

7. Given that the system $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 2 \\ -4 & 6 \end{pmatrix}\mathbf{Y}$ has eigenvalues $\lambda = 4 \pm 2i$, and an eigenvector corresponding to $\lambda = 4 + 2i$ is $\begin{pmatrix} 1 \\ 1+i \end{pmatrix}$, write a general solution to the system.

a solution would be:

$$\hat{Y} = e^{(4+2i)t}\begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$

$$= e^{4t} e^{2it}\begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$

$$= e^{4t}(\cos 2t + i\sin 2t)\begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$

$$= e^{4t}\begin{pmatrix} \cos 2t + i\sin 2t \\ \cos 2t + i\cos 2t + i\sin 2t - \sin 2t \end{pmatrix}$$

$$\hat{Y} = e^{4t}\begin{pmatrix} \cos 2t \\ \cos 2t - \sin 2t \end{pmatrix} + ie^{4t}\begin{pmatrix} \sin 2t \\ \cos 2t + \sin 2t \end{pmatrix}$$

Excellent!

By Complex Miracle Theorem:

$$\hat{Y} = Ae^{4t}\begin{pmatrix} \cos 2t \\ \cos 2t - \sin 2t \end{pmatrix} + Be^{4t}\begin{pmatrix} \sin 2t \\ \cos 2t + \sin 2t \end{pmatrix}$$

is a general solution

8. Find a general solution to the equation $y'' + 9y = 2\sin 3t$.

Let $y = e^{st}$ for homogeneous solution,

$$y' = se^{st} \quad , \quad y'' = s^2 e^{st}$$

$$s^2 e^{st} + 9e^{st} = 0 \quad \Rightarrow \underline{s = \pm 3i}$$

Then $\underline{y_h = e^{3it} \Rightarrow k_1 \cos 3t + k_2 \sin 3t}$

Now, let $\underline{y_p = At\cos 3t + Bt\sin 3t}$

$$y_p' = A\cos 3t - 3At\sin 3t + B\sin 3t + 3Bt\cos 3t$$

$$y_p'' = -3A\sin 3t - 3A\sin 3t - 9At\cos 3t + 3B\cos 3t + 3B\cos 3t - 9Bt\sin 3t$$

$$= -6A\sin 3t + 6B\cos 3t - 9At\cos 3t - 9Bt\sin 3t$$

Then

$$-6A\sin 3t + 6B\cos 3t - 9A\cancel{t\cos 3t} - 9B\cancel{t\sin 3t} + 9A\cancel{t\cos 3t} + 9B\cancel{t\sin 3t} = 2\sin 3t$$

So, $-6A = 2$, $6B = 0$ $\Rightarrow \underline{A = -\frac{1}{3}}, \; B = 0$

Thus, $\boxed{y = k_1 \cos 3t + k_2 \sin 3t - \frac{1}{3} t\cos 3t}$

Excellent!

9. Suppose that $\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$ has a coefficient matrix of the form $A = \begin{pmatrix} 0 & 1 \\ a & 2b \end{pmatrix}$. For what values of a and b will the origin be a source?

$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$ The origin will be a sorce if both λ are positive and real, so T must be positive and $T^2 - 4D$ must be positive and less than T^2, so D must be positive and less than $T^2/4$

We have $T = 0 + 2b = 2b$, and $D = (0)(2b) - (a)(1) = -a$

so b must be positive, a must be negative

and $D < T^2/4$ so $-a < (2b)^2/4$ $a > -b^2$

so our conditions are:

$\boxed{\begin{array}{l} a < 0 \\ b > 0 \\ a > -b^2 \end{array}}$

Outstanding.

10. Show that if λ is an eigenvalue of a matrix $\mathbf{A}$, with corresponding eigenvector $\mathbf{v}$, then $\mathbf{Y} = e^{\lambda t}\mathbf{v}$ is a solution to $\dfrac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$.

well, if $\hat{Y} = e^{\lambda t}\hat{V}$, $\frac{d\hat{Y}}{dt} = \lambda e^{\lambda t}\hat{V}$, so lets plug in the values and see if it works:

$$\left.\begin{aligned}\frac{d\hat{Y}}{dt} &= \lambda e^{\lambda t}\hat{V} \\ &= \lambda \cdot \hat{Y} \\ &= \hat{A}\cdot\hat{Y}\end{aligned}\right\}$$ by definition of eigenvectors, $\lambda\hat{Y} = \hat{A}\hat{Y}$

so, since we plug it in, and it worked,

$\hat{Y} = e^{\lambda t}\hat{V}$ is a solution $\square$ Nicely done!